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PRINCIPLES OF THE MACHINE ARITHMETIC OF COMPLEX NUMBERS.(U)
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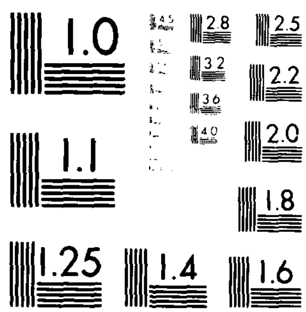

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PRINCIPLES OF THE MACHINE ARITHMETIC OF COMPLEX NUMBERS

by

I. Ya. Akushskiy, V. M. Amerbayev, I. T. Pak

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~~By:~~ I. Ya. Akushskiy, V. M. Amerbayev, I. T. Pak

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U. S. BOARD ON GEOGRAPHIC NAMES TRANSLITERATION SYSTEM

Block	Italic	Transliteration	Block	Italic	Transliteration
А а	*А а*	A, a	Р р	*Р р*	R, r
Б б	*Б б*	B, b	С с	*С с*	S, s
В в	*В в*	V, v	Т т	*Т т*	T, t
Г г	*Г г*	G, g	У у	*У у*	U, u
Д д	*Д д*	D, d	Ф ф	*Ф ф*	F, f
Е е	*Е е*	Ye, ye; E, e*	Х х	*Х х*	Kh, kh
Ж ж	*Ж ж*	Zh, zh	Ц ц	*Ц ц*	Ts, ts
З з	*З з*	Z, z	Ч ч	*Ч ч*	Ch, ch
И и	*И и*	I, i	Ш ш	*Ш ш*	Sh, sh
Й й	*Й й*	Y, y	Щ щ	*Щ щ*	Shch, shch
К к	*К к*	K, k	Ъ ъ	*Ъ ъ*	"
Л л	*Л л*	L, l	Ы ы	*Ы ы*	Y, y
М м	*М м*	M, m	Ь ь	*Ь ь*	'
Н н	*Н н*	N, n	Э э	*Э э*	E, e
О о	*О о*	O, o	Ю ю	*Ю ю*	Yu, yu
П п	*П п*	P, p	Я я	*Я я*	Ya, ya

*ye initially, after vowels, and after ъ, ь; e elsewhere.
When written as ё in Russian, transliterate as yё or ё.

RUSSIAN AND ENGLISH TRIGONOMETRIC FUNCTIONS

Russian	English	Russian	English	Russian	English
sin	sin	sh	sinh	arc sh	$\sinh^{-1}$
cos	cos	ch	cosh	arc ch	$\cosh^{-1}$
tg	tan	th	tanh	arc th	$\tanh^{-1}$
ctg	cot	cth	coth	arc cth	$\coth^{-1}$
sec	sec	sch	sech	arc sch	sech^{-1}
cosec	csc	csch	csch	arc csch	csch^{-1}

Russian	English
rot	curl
lg	log

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PRINCIPLES OF THE MACHINE ARITHMETIC OF COMPLEX NUMBERS.

I. Ya. Akushskiy, V. M. Amerbayev, I. T. Pak.

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In this work are studied the machine algorithms of the systems of numeration (both positional and nonpositional) with the composite bases/bases.

The results of this work can be of interest both for developers of TsVM, which lead the search for fundamentally new constructive solutions of organizing AU TsVM, and for the developers of elements/cells with the specific character of the arithmetic of complex numbers in numeration systems with the composite bases/bases.

The book is intended for engineers, scientific workers, graduate students and students, who specialize in the area of computational technology.

33 tables, 52 illustrations, 15 references.

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PREFACE.

The rates of development of science and technology, depth and comprehensiveness of the conducted scientific research lead to the setting of ever more complex problems whose solution is necessary for the national economy. It requires the growth of the productivity of electronic computers. Even now on 1970-1980 is forecast the creation of computational means by the productivity of 10^9 operations per second.

If we clear the questions, connected with the development of the new methods of the numerical solution of tasks, programming and the creation of new machine languages, then the ways of further perfection of computational means are examined/scanned both in the improvement in the engineering and technical basis and in the development of structure, logic and organization of computers.

The first path represents the sufficiently prolonged process which includes at the initial stage the discovery/opening the physical principles of the construction of elements/cells with the high speed operation of changeover of one state in another, at the

following stage - development of new efficient technological methods (in the relation to the possibilities of mass production and economically) and finally at the completing stage - conversion of those existing and introduction of new production enterprises. This is the very heavy path, which requires considerable time and enormous means.

Computer technology in this plan/layout passed already the series/row of stages. With the advent of relay technology purely mechanical computational means passed into the electromechanical ones. Then, when arose vacuum electronic engineering, began to be constructed the electron-tube electronic computers, which together with radio electronics completed transition to the semiconductor technology.

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At present computational means prepare to do the following step/pitch - to pass to microelectronic technology and integral chart technology.

Another matter is the alternate path. It does not require any technical and technological improvements, but is based on the creation of new theoretical concepts in representation and

information processing in the machine, in the construction of its structure and logic and in the organization of the very process of the solution of problems in the machines.

At the basis of all methods created by this method, lies/rests the idea very simple and at the same time very complicated - the deparallelization of operations. Simple it because logically it appears, as soon as one machine not in the state to manage the data by the capacity of computational operations and, that means it is necessary to draw other machines and to distribute the work between them for its simultaneous and parallel execution. This is, in the known sense, the principle of the organization of any mass production when different parts of article in parallel are made on different equipment and only at the final stage they are gathered into the article in the final form. Is complicated this idea by the fact that the solution far not of any problem can be dismembered to the parts, which allow/assume parallel execution in different machines. In the majority of the cases the algorithms of the solution carry especially consecutive character and only by complicated artificial methods it is possible to produce necessary separation, spending in this case the significant part of the total productivity of machine to the realization of this separation and for the subsequent connecting/fitting of all parts of the solution between themselves. In spite of such very essential difficulties in the organization of

the multiple operation of different machines above the solution of large problems, the creation of multimachine computing systems is at present the important method of the satisfaction of the necessities for the high productivity.

Is very promising and is fruitful in the practical plan/layout the deparallelization not of the algorithms of the solution of problem, but deparallelization on the "microlevel" of the algorithms of the execution of elementary arithmetic operations. Here it was required to find a theoretical-numerical basis, which is determining the methods of the separation of a number into the individual parts, which allow/assume their independent and parallel processing. This basis was the theory of comparison.

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Relying on fundamental concepts and positions of this theory, it was possible to construct the original numeration system in the residual classes, in which numbers are represented by their deductions on the basis of mutually simple moduli/modules - the basis of system, and the rational operations on numbers are conducted independent of the appropriate deductions of these numbers individually. Numeration system in the residual classes underwent development both in the theoretical and pr' ctical ' an/layout and it was used as basis for

the construction of high-productivity computers.

Meanwhile the ideas and the concepts of the system of residual classes proved to be such attractive that caused the tendency to widen sphere their applications/appendices and to generalize them in different directions.

Since in the system of residual classes are absent the evident interbit/interbyte connections, then natural was generalization in the direction of the creation of the general theory of nonpositional systems, which would contain all possible similar numeration systems. This theory is developed/processed by the works of the series/row of Soviet scientists and it came to light/detected/exposed the already very interesting numeration system, which possess different specific properties, important for the efficient practical realization.

Another not less important direction - generalization of the system of residual classes to the objects of more complicated nature than the region of real numbers. Work in this direction was conducted in the institute of mathematics and mechanics of AN KazSSR.

The following in its complexity is the region of complex numbers. The construction of numeration system in the residual classes in complex domain proved to be possible on the basis of the

Gaussian theory complex integers, and Gaussian idea of the isomorphism between the composite deductions of a number on the composite modulus/module and his real deductions according to the norm of this modulus/module made it possible to wholly replace the integrated numeration system with real and thereby virtually it created the possibility to work in the real region with complex numbers as a whole without their separation into real and complex domains.

It is difficult to overestimate the practical value which has this possibility for solving the tasks, formed/shaped in the terms of complex quantities. The algorithms of the solution of problems considerably are simplified, since in them is absent this element/cell, as isolation of the independently real and alleged parts of the values and their couplings, and the productivity of the solution of problems in this case sharply grows.

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The possibility to use with a complex number (or flat/plane vector) as by the elementary inseparable object promises the interesting prospects for the creation of the new methods of the numerical solution of many important tasks. Over the long term - transposition of this method to the three-dimensional/space multidimensional

vectors, and also to the elements/cells of some function spaces.

The book contains the results of research by construction of machine arithmetic in complex domain in the described above direction.

In the first three chapters, being based on K. F. Gauss's works on the theory of biquadratic deductions [8], are studied basic questions of the division theory, theory of comparisons, theory of the indices complex integers. Special attention is paid to the analysis of the concept of the full/total/complete system of deductions on the composite moduli/modules and to the analysis of the tables of modular arithmetic.

The fourth chapter is dedicated to the study of the properties of the positional numeration systems with the composite bases/bases.

The fifth chapter contains the generalization of the theory of the numeration system of residual classes to complex domain and the development of machine algorithms in these numeration systems. By the introduction of special coding are examined the methods of the abridgement of table of modular addition and multiplication. Are studied the operations of shortening and expanding the range.

The book will be of interest both for the mathematicians, who work in the region to the theory of mathematical machines and their use/application, as well as for the development engineers of the digital computers, which lead the search for the fundamentally new ways of organizing the arithmetic units of computers and increase in their efficiency.

The authors express a deep appreciation to E. I. Al'zamarova for the shown/rendered help in the calculation of tables and the formulation of the manuscript.

The authors.

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Chapter 1.

DIVISION THEORY IN A RING OF COMPLEX INTEGERS.

§1. Ring of complex integers.

Let us designate through Γ the set of all complex numbers $a+bi\,(\sqrt{-1}=i)$, real and alleged parts of which are integers. Such numbers are called complex integers or whole gaussian numbers (sometimes by Gaussians) in the honor of K. F. Gauss, who for the first time in detail studied the arithmetic complex integers in the famous research "Theory of biquadratic deductions" [8]. Subsequently complex integers we will briefly write/record ts.k.ch., whole real numbers - ts.v.ch.

Since for any ts.k.ch. $u, w, v \in \Gamma$ is fulfilled

$$u+w \in \Gamma,$$
$$uw \in \Gamma,$$
$$-u \in \Gamma,$$

moreover

$$u+w=w+u,$$
$$u \cdot w = w \cdot u,$$
$$u(w+v)=uw+uv,$$

then set of ts.k.ch. Γ is carried out commutative ring.

Ring G does not have zero divisors, which is equivalent to the assertion: from condition u•w=0 it follows that, at least, one of the cofactors is equal to 0.

Let us designate through P the set of all complex numbers whose real and alleged parts are rational numbers.

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Since sum, difference, product, quotient (if divider/denominator it is different from 0) two numbers of P again belongs P, then P is field. Ring G is contained in field P, but it is not it really is easy to construct quotient of two ts.k.ch. which is not ts.k.ch.

Any number of P can be represented in the form of relation of two ts.k.ch.; therefore P call the quotient field of ring Γ.

Norm of ts.k.ch. z=a+bi is called the square modulus of this number. For the norm is built-in the following symbol:

$$\| z \| = a^2 + b^2. \qquad (1.1)$$

It is obvious,

$$\| z \cdot w \| = \| z \| \cdot \| w \| . \qquad (1.2)$$

§2. Divisibility in the ts.k.ch. ring G

Since

$$\frac{w}{z}=\frac{a+bi}{c+di}=\frac{ac+bd}{c^2+d^2}+\frac{bc-ad}{c^2+d^2}i, \quad (2.1)$$

that divisibility "completely" of ts.k.ch. w to ts.k.ch. z is equivalent to divisibility of ts.v.ch.: as+bc, bc-ad on $\|z\|=a^2+b^2$, or

$$\begin{aligned} ac+bd &\equiv 0 \pmod{\|z\|}; \\ bc-ad &\equiv 0 \pmod{\|z\|}. \end{aligned} \quad (2.2)$$

In such cases it is accepted to write z/w (it is read: "z divides w"), otherwise: zxw. R That the relation of divisibility in ring G possesses the following properties: 1) z/0, 1/z, z/z for any z: 2) 0xz for any z(z≠0): 3) from z/w, w/u it follows z/u: 4) from z/w, u/v follows zu/wv: 5) from zu/wu, u≠0 follows z/w: 6) from z/w_i $(1\leqslant i\leqslant n)$ follow

$$z/\sum_{i=1}^{n}u_i w_i$$

for any $u_1, u_2, \ldots, u_n$.

If ts.k.ch. $\varepsilon/1$, then it is called the divider/denominator of unity. Hence it follows that the divider/denominator of unity in ring G are all those numbers, the reciprocals numbers to which there are ts.k.ch.: $\varepsilon^{-1}=\frac{1}{\varepsilon}$ and $\varepsilon^{-1}\in\Gamma$.

Since $\varepsilon=(\varepsilon^{-1})^{-1}$, that ε^{-1} is also the divider/denominator of unity. Further, since $\varepsilon\cdot\varepsilon^{-1}=1$, that $\|\varepsilon\cdot\varepsilon^{-1}\|=\|\varepsilon\|\cdot\|\varepsilon^{-1}\|=1$.

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Since the norm of ts.k.ch. must be ts.v.ch., latter/last equality is possible only when $\|\varepsilon\|=1$ and $\|\varepsilon^{-1}\|=1$. Thus, ts.k.ch. z then only then it is the divider/denominator of unity when $\|z\|=1$.

Let z=a+bi and $\|z\|=1$. In that case

$$a^2+b^2=1.$$

Latter/last equation has the following solutions in the integers:

$$a=\pm 1,\ b=0;\ a=0,\ b=\pm 1.$$

In other words, in ring G four dividers/denominators of unity:

$$+1;\ -1;\ +i;\ -i.$$

Ts.k.ch. that differ from each other in terms of the multiplier, equal to the divider/denominator of unity, are called associated. Thus, the numbers, associated with ts.k.ch. a+bi, will be

with

$$-a-bi;\ -b+ai;\ b-ai,$$

and, ts.k.ch. a-bi (conjugated/combined to number a+bi) - the numbers

$$-a+bi;\ b+ai;\ -b-ai.$$

A somewhat special position occupy the associated numbers

$$\begin{matrix} a+ai; & -a-ai; & -a+ai; & a-ai; \\ b; & -b; & bi; & -bi; \end{matrix}$$ (a, b — ts.v.ch.),

which differ in terms of the fact that are connected in their group the conjugated/combined numbers.

The relation of fissionability is invariant to the dividers/denominators of unity in the sense that it is not broken, if given ts.k.ch. are replaced with any others, with them those associated, i.e., if $\varepsilon_1, \varepsilon_2$ - dividers/denominators of unity and z/w, then $z\varepsilon_1 / w\varepsilon_2$.

Actually/really, since $\varepsilon_1 / 1$ and $1 / \varepsilon_2$, that $\varepsilon_1 / \varepsilon_2$. Further, since z/w and $\varepsilon_1 / \varepsilon_2$, that $z\varepsilon_1 / w\varepsilon_2$.

Example. 3+2i divides 4+7i, i.e., (4+7i)/(3+2i)=2i.

Also (3+2i)(-1) divides (4+7i)i, i.e., (-7+4i)/(-3-2i)=1-2i.

§3. Prime numbers of Gauss.

The dividers/denominators of unity and all numbers, associated with ts.k.ch. z, are called trivial dividers/denominators of ts.k.ch. z.

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Ts.k.ch. are called composite/compound, if there are nontrivial dividers/denominators of this number, and simple - otherwise.

Prime numbers we will call the prime numbers of Gauss (p.ch.G.) in contrast to the term "prime numbers (p.ch.), difference from the term "prime numbers (p.ch.), which relate to ts.v.ch.

It is obvious, any composite/compound natural number is composite/compound in ring G. However, not any p.ch. is p.ch.G.

Actually it is known that p.ch. of form $4n+1$ are decomposed/expanded into the sum of squares; therefore such numbers can be represented in the form of the product of two of those conjugated/combined ts.k.ch. as, for example, $2=(1+i)(1-i)$, $5=(2+i)(2-i)$.

But p.ch. of form $4n+3$ are p.ch.G. Actually/really, let p.ch. $p=4n+3$ and $p=(a+bi)(c+di)$, then $\overline{p}=(a-bi)\cdot(c-di)$ and, therefore,

$$p^2=(a^2+b^2)(c^2+d^2).$$

Since p - p.ch. then $a^2+b_2=c^2+d^2=p$. Further, in view of oddness p numbers a and b have opposite parity. Let $a=2k$, $b=2m+1$, then

$$p=a^2+b^2=4k^2+4m^2+4m+1,$$

i.e. $p \equiv 1 \pmod 4$, whereas $p \equiv 3 \pmod 4$.

They arrived at the contradiction.

Theorem 3.1. So that ts.k.ch. z=a+bi (a, b≠0) would be p.ch.G. is necessary and it is sufficient so that $\|z\|$ there would be p.ch.

Proof. Let $\|z\|=p$ - p.ch. and let $z=u \cdot w$, then

$$p = \|u\| \cdot \|w\|.$$

Since norms of ts.k.ch. are ts.v.ch., then one of the multipliers $\|u\|$, $\|w\|$ must be equal to 1. If $\|w\|=1$, then w - divider/denominator of unity. Thus, z does not have other dividers/denominators, except trivial ones.

Conversely, let $\|z\|=a^2+b^2$ be composite number and let $p/\|z\|$ (p - p.ch.). Are possible two cases.

Case 1. p - p.ch. of form 4n+1, then $p=\alpha^2+\beta^2$; let us show that ts.k.ch. z is divided into $\alpha+i\beta$ or on $\alpha-i\beta$.

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We have equalities

$$\frac{a+bi}{\alpha+i\beta} = \frac{a\alpha+b\beta}{p} + \frac{b\alpha-a\beta}{p} i;$$

$$\frac{a+ib}{\alpha-i\beta} = \frac{a\alpha-b\beta}{p} + \frac{b\alpha+a\beta}{p} i.$$

Hence we obtain the identities

$$\|z\|\cdot p=(a\alpha+b\beta)^2+(b\alpha-a\beta)^2;$$
$$\|z\|\cdot p=(a\alpha-b\beta)^2+(b\alpha+a\beta)^2.$$

Further, since

$$(a\alpha+b\beta)\cdot(a\alpha-b\beta)=a^2p-\beta^2\|z\|,$$

the either $p/(a\alpha+b\beta)$, or $p/(a\alpha-b\beta)$. In the first case from the first identity it follows that $p/(b\alpha-a\beta)$ and therefore $(\alpha+i\beta)/(a+bi)$: in the second case from the second identity follows that $p/(b\alpha+a\beta)$ and therefore $(\alpha-i\beta)/(a+bi)$.

Case of 2. p - p.ch. of form 4n+3; in that case number $\|z\|=a^2+b^2$ is divided into p if and only if p/a and p/b, i.e., number a+bi - is composite/compound.

Thus, it is shown that if the norm of ts.k.ch. z is composite/compound, then number itself z - composite/compound.

Corollary 1. Norm $\|z\|$ p.ch.G. z is either p.ch. or square p.ch.; in this case in the first case p.ch.G. has different from zero real and alleged parts, and secondly - p.ch.G. with an accuracy to the dividers/denominators of unity coincides with p.ch.

Corollary 2. Set p.ch.G. can be classed as follows:

(1) Тип п. ч.	(2) Простые числа Гаусса (п. ч. Г.)				(3) Норма п. ч. Г.
0	1	i	-1	$-i$	$\lVert z \rVert = 1$
1	$1+i$	$-1+i$	$-1-i$	$1-i$	$\lVert z \rVert = 2$
2	$\alpha+i\beta$	$-\beta+i\alpha$	$-\alpha-i\beta$	$\beta-i\alpha$	$\lVert z \rVert = p$—п. ч. $p\equiv 1 \pmod 4$
3	p	ip	$-p$	$-ip$	$\lVert z \rVert = p^2$ $p\equiv 3 \pmod 4$

Key: (1). Type of p.ch. (2). Prime numbers of Gauss (p.ch.G.). (3). Norm of p.ch.G.

§4. Properties of divisibility in ring G

Theorem 4.1. For any ts.k.ch. z and p there are ts.k.ch. t and r such, that

$$z = tp + r, \qquad (4.1)$$

moreover

$$\lVert r \rVert < \lVert p \rVert. \qquad (4.2)$$

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Proof. Let $z=a+bi$ and $p=\alpha+i\beta$. Let us consider the number

$$\frac{a+bi}{\alpha-i\beta} = \frac{a\alpha+b\beta}{\lVert p \rVert} + i\,\frac{a\beta-\alpha b}{\lVert p \rVert} = \xi + i\eta.$$

Let τ and σ - respectively nearest ts.ch. to the real and alleged parts of the fraction, i.e.

$$|\tau-\xi| \leqslant \frac{1}{2}; \quad |\sigma-\eta| \leqslant \frac{1}{2}. \qquad (4.3)$$

We form ts.k.ch. $t=\tau+i\sigma$, then

$$\left\lVert \frac{z}{p} - t \right\rVert = \left(\frac{a\alpha-b\beta}{\lVert p \rVert} - \tau\right)^2 + \left(\frac{a\beta-\alpha b}{\lVert p \rVert} - \sigma\right)^2 \leqslant \frac{1}{2}.$$

Let us assume r=z-tp, then z=tp+r, moreover

$$\|r\| = \left\|\left(\frac{z}{p}-t\right)p\right\| = \left\|\frac{z}{p}-t\right\|\cdot\|p\| \leqslant \frac{1}{2}\|p\| < \|p\|.$$

The property of ts.k.ch. expressed by the proved theorem, means that ring G is Euclidian ring [7], for which remains valid the euclidean algorithm. Euclidean algorithm for ts.k.ch. will be described below.

Relationship/ratio (4.1) is generalized the relation of divisibility, in this case, as usual, number z is called dividend, p - by divider/denominator (modulus/module), t - by the partial quotient, r - by remainder/residue.

The relatively partial quotient (t) and remainder/residue (r) theorem 4.1 claims their only existence. Requirement (4.2), generally speaking, does not provide the uniqueness of pair t and r.

Example. $p = 3-2i$, $\quad \|p\| = 13.$

$$\begin{aligned} 5+4i &= 1\cdot p + (2+2i), & \|2+2i\| &= 8 < \|p\|, \\ 5+4i &= 2p + (-1), & \|-1\| &= 1 < \|p\|, \\ 5+4i &= (2+i)p + (1-3i), & \|1-3i\| &= 10 < \|p\|. \end{aligned}$$

A question about the conditions of the uniqueness of representation of ts.k.ch. in the form (4.1) has important value for the machine arithmetic of ts.k.ch. and it will be examined below.

For ts.k.ch. by the usual method is built-in the concept of the

greatest common divisor (NOD). Ts.k.ch. d is NOD of ts.k.ch. $z_1, z_2, \ldots, z_n$; $(d=(z_1, z_2, \ldots, z_n))$, if

1) d is the common divisor of all numbers $z_1, z_2, \ldots, z_n$;

2) d are divided into any common divisor of numbers $z_1, z_2, \ldots, z_n$.

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Ts.k.ch. $z_1, z_2, \ldots, z_n$ is mutually simple, if $(z_1, z_2, \ldots, z_n)=1$.

Theorem 4.2. For any final set of ts.k.ch. $z_1, z_2, \ldots, z_n$ there exists with an accuracy to the dividers/denominators of unity only NOD. In this case $d=(z_1, z_2, \ldots, z_n)$ when and only when

$$\begin{gathered} z_k = dw_k \\ d = \sum_{k=1}^{n} u_k z_k \end{gathered} \qquad 1 \leq k \leq n \qquad (4.4)$$

with some ts.k.ch. w_k, u_k $(1 \leq k \leq n)$.

Proof. On the basis of ts.k.ch. $z_1, z_2, \ldots, z_n$ we form set of ts.k.ch. L by means of all possible combinations of form $\sum_{k=1}^{n} \lambda_k z_k$.

Set L - subring, since from $a, b \in L$ it follows

$$a \pm b \in L, \ a, b \in L,$$

$$a \pm b = \sum_{k=1}^{n} (\lambda_k' \pm \lambda_k'') z_k \in L,$$

$$a \cdot b = \sum_{k=1}^{n} (\lambda_k' \cdot \lambda) z_k \in L.$$

Let us consider the norms of all ts.k.ch. of subring L. Since from $\|z\|=0$ it follows $z=0$, then among different from zero ts.k.ch. sets L there is, at least, one ts.k.ch. d with the smallest value of norm.

Let us show that any ts.k.ch. $z \in L$ is divided into d. Let $z=q \cdot d+r$, where $\|r\|<\|d\|$, Since $d, z \in L$. that $r=z-qd \in L$. But ts.k.ch. d has on set L minimum norm; therefore from conditions $r \in L$ and $r \leqslant d$ it follows that r=0.

Further, since $z_k \in L (1 \leqslant k \leqslant n)$, first $d \cdot z_k (1 \leqslant k \leqslant n)$, i.e. carried out condition (4.3); since $d \in L$, that will be located such ts.k.ch. $u_1, u_2, \ldots, u_n$ that

$$d = \sum_{k=1}^{n} u_k z_k,$$

i.e. is satisfied condition (4.4). But any ts.k.ch. d, which satisfies conditions (4.3), (4.4), is NOD of numbers $z_1, z_2, \ldots, z_n$.

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Actually/really, from (4.3) it follows that d is the common divisor of numbers $z_1, z_2, \ldots, z_n$, and from (4.4) it follows that any common divisor of numbers $z_1, z_2, \ldots, z_n$ is divider/denominator of ts.k.ch. d.

It remains to show that NOD is determined by

relationships/ratios (4.3), (4.4) with an accuracy to the dividers/denominators of unity. Let it be ε - divider/denominator of unity, then (4.3) and (4.4) it is possible to present in the form

$$z_k = (\varepsilon \cdot d) \cdot (w_k \cdot \varepsilon^{-1}),$$

$$\varepsilon \cdot d = \sum_{k=1}^{n} (\varepsilon u_k) \cdot z_k,$$

i.e. ts.k.ch. $\varepsilon \cdot d$ also is NOD of elements/cells, $z_1, z_2, \ldots, z_n$.

Corollary 1. In relationship/ratio (4.3) numbers $w_1, w_2, \ldots, w_n$ are mutually simple, in this case

$$\sum_{k=1}^{n} u_k w_k = 1.$$

Corollary 2. If $w_1, w_2, \ldots, w_n$ - is mutually simple and $z_k = dw_k \, (1 \leqslant k \leqslant n)$, then

$$d = (z_1, z_2, \ldots, z_n).$$

Corollary 3. If z=a+bi and (a, b)=1, then $(z, \bar{z})=1$.

In fact, let us assume that $(z, \bar{z}) = \rho$ $(\rho \neq 1)$. Since $(\bar{z}, \bar{z}) = \bar{\rho}$ and $(z, \bar{z}) = (\bar{z}, z)$, then ρ - ts.v.ch. Hence we consist: $z=\rho(a_1+b_1 i)$, which contradicts condition (a, b)=1.

Theorem 4.3 (euclidean algorithm). NOD of ts.k.ch. z, w can be found with the help of the algorithm, described below.

Proof. If w=0, then (z, w)=z; but if $w \neq 0$, then is realized the process of consecutive indexing on the diagram

$$
\begin{array}{ll}
z = wq_1 + r_1 & \| r_1 \| < \| w \| \\
w = r_1 q_2 + r_2 & \| r_2 \| < \| r_1 \| \\
r_1 = r_2 q_3 + r_3 & \| r_3 \| < \| r_2 \| \\
\cdots\cdots & \cdots\cdots \\
r_{s-2} = r_{s-1} q_s + r_s & \| r_s \| < \| r_{s-1} \| \\
r_{s-1} = r_s q_{s+1} & r_{s+1} = 0
\end{array}
\qquad (4.5)
$$

Since all $\| r_i \|$ - whole non-negative numbers and

$$\| w \| > \| r_1 \| > \| r_2 \| > \| r_3 \| > \ldots,$$

that the described process of consecutive indexing is final and at certain s+1- step/pitch we will obtain $r_{s+1} = 0$.

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Let us show that ts.k.ch. r_s will be unknown NOD. From equalities (4.5), examining/scanning them from bottom to top, we see that

$$r_s / r_{s-1},\ r_s / r_{s-2},\ r_s / r_{s-3}, \ldots,\ r_s / w,\ r_s / z,$$

i.e. r_s is common divisor of ts.k.ch. z and w. Let now ζ - any divider/denominator of numbers z and w, then, examining/scanning

equalities (4.5) also downward, we note that

$$\zeta / r_1, \zeta / r_2, \zeta / r_3, \ldots, \zeta / r_s,$$

i.e. $r_s = (z, w)$.

Observation. Using a euclidean algorithm, it is possible to find NOD for any final set of ts.k.ch. $z_1, z_2, \ldots, z_n$ from the diagram

$d_1 = (z_1, z_2),\ d_2 = (d_1, z_3),\ d_3 = (d_2, z_4),\ \ldots,\ d = (d_{n-2}, z_n) = (z_1, z_2, \ldots, z_n).$

Example.

a) To find NOD of ts.k.ch. (1+3i, 1-3i):

$$1 + 3i = (1 - 3i)(-1 + i) + (-1 - i),\quad \| -1 - i \| = 2 <$$

$$< \frac{1}{2} \| 1 - 3i \| = \frac{1}{2} \cdot 10,$$

$$1 - 3i = (-1 - i)(1 + 2i) + 0,$$

$$(1 + 3i, 1 - 3i) = -1 - i.$$

b) to find NOD of ts.k.ch. $(-3 + 29i, 3 + 11i, 10 + 4i)$:

$$-3 + 29i = (3 + 11i)(2 + i) + 2 + 4i,\quad \| 2 + 4i \| = 20 < \| 3 + 11i \| = 130,$$

$$3 + 11i = (2 + 4i) 2 + (-1 + 3i),\quad \| -1 + 3i \| = 10 < \| 2 + 4i \| = 20,$$

$$2 + 4i = (-1 + 3i)(1 - i) + 0,$$

whence (-3+29i, 3+11i)=1-i. Further

$$10 + 4i = (1 - i)(3 + 7i) + 0,$$

therefore

$$(-3 + 29i, 3 + 11i, 10 + 4i) = 1 - i.$$

Theorem 4.4. If $w / z\zeta$ and $(w, \zeta) = 1$, then w/z.

Proof. Since $(w, \zeta) = 1$, then there are ts.k.ch. u_1, u_2 such, that

$$u_1 w + u_2 \zeta = 1.$$

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Multiplying this equality on z, we will obtain $z=u_1wz+u_2(\zeta\cdot z)$. Since $w/\zeta\cdot z$ and w/u_1zw, then w/z.

Theorem 4.5. If z - any ts.k.ch. and p - p.ch.G. then for (p, z)=1 is necessary and sufficiently, in order to $p\nmid z$.

Proof. Let (p, z)=d; therefore d/p. Since p - p.ch.G. the ts.k.ch. d can be either the number, associated with p or by the divider/denominator of unity. But d cannot be the number, associated with p, since otherwise they would have, that $p \sim d$. but since d/z, then would be obtained that p/z. Thus, $\|d\|=1$, i.e., d - the divider/denominator of unity and, therefore, p, z is mutually simple.

Conversely, if (p, z)=1, then $p\nmid z$, since from $p\mid z$, and p/p would follow that p/1 that it contradicts so that $\|p\|\neq 1$.

Theorem 4.6. If product of ts.k.ch. $z_1\cdot z_2\cdot\ldots\cdot z_n$ is divided into p.ch.G. p, then at least one of the cofactors is divided into p.

Proof. By induction on n theorem can be reduced to product of

two ts.k.ch. $z_1 \cdot z_2$. If p/z_1, then according to theorem 4.5 $(p, z_1)=1$, then according to theorem 4.4 p/z_2.

Theorem 4.7. Any ts.k.ch. z, different from 0 and dividers/denominators of unity, is decomposed/expanded into the product p.ch.G. and besides by only form with an accuracy to to the order of cofactors and dividers/denominators of unity.

Proof. Let z - arbitrary ts.k.ch. Let us demonstrate at first the existence of its disintegration on p.ch.G.

If $\|z\|=1$ and z is not factorable, different from the numbers, associated with z, then z - p.ch.G.

Let z have the nontrivial dividers/denominators $z=c_1 \cdot c^{(1)}_1$. At least one of them (let us say $c^{(1)}_1$) it has the nontrivial dividers/denominators $c^{(1)}_1=c_2 c^{(2)}_2$. Otherwise the existence of expansion z on p.ch.G. is proved. As a result of the consecutive use/application of these reasonings we will obtain chain/network of ts.k.ch. $c^{(1)}_1$, $c^{(2)}_2$, $c^{(3)}_3$, ..., moreover $\|z\| > \|c_1^{(1)}\| > \|c_2^{(2)}\| > \|c_3^{(3)}\| > \ldots$

Since norms of ts.k.ch. are integers, then through a finite number of steps/pitches this chain/network is broken, i.e., each such chain/network will lead to that decomposed ts.k.ch. (p.ch.G.). This

proves to be the existence of the disintegration of any ts.k.ch. on p.ch.G.

Let us demonstrate the uniqueness of disintegration. If ts.k.ch. z is p.ch.G. then the uniqueness of its disintegration with an accuracy to the dividers/denominators of unity is obvious. Let us assume by the induction that is valid the uniqueness of disintegration with a number of simple multipliers not more than n-1. Let further ts.k.ch. w have two expansions (with a number of multipliers, equal n): $w=p_1p_2\dots p_n=q_1q_2\dots q_s$, where all p_i, p_j - p.ch.G.

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Since w is divided into p_1, then one of the cofactors $q_1, q_2, \dots, q_n$ is divided into p_1. Let this be q_1. Since q_1 - p.ch.G., then $q_1=\varepsilon_1 p_1$, where $\|\varepsilon_1\|=1$. Shortening to p_1 initial equality, we will obtain ts.k.ch. with the number of cofactors, which does not exceed n-1. Since by hypothesis for a number of cofactors not above n-1 uniqueness of disintegration on p.ch.G. is accurate, then n-1=s-1.

Thus, with the adequate/approaching numbering we will have $q_i=\varepsilon_i p_i$, where $\|\varepsilon_i\|=1$.

Example. Ts.k.ch. 109+142i uniquely is represented in the form

$$109+142i=(2+i)(3+2i)(4+i)(5-2i).$$

The concept of the least common multiple (NOK) of several ts.k.ch. is built-in by the usual method:

z is NOK of ts.k.ch. $w_1, w_2, \ldots, w_n$, if z is common multiple of numbers $w_1, w_2, \ldots, w_n$ and divides any common multiple of them.

Decomposing/expanding each ts.k.ch. $w_1, w_2, \ldots, w_n$ on p.ch.G. we form ts.k.ch. z, equal to the product of all p.ch.G. of entering the disintegration at least of one of the numbers w_i. It is obvious that thereby NOK are determined unambiguously with accuracy to the dividers/denominators unity.

Theorem 4.8. If ts.k.ch. z is divided into each ts.k.ch. $p_1, p_2, \ldots, p_n$ and these latter pairwise are mutually simple, then $p_1, p_2, \ldots, p_n / z$.

Proof. The simple multipliers of each of the numbers p_k enter into z, in this case any pair of numbers p_k, p_l does not have common factors. Therefore

$$p_1, p_2, \ldots, p_n / z.$$

Corollary. If ts.k.ch. $p_1, p_2, \ldots, p_n$ is pair-wise mutually simple, then the NOK of numbers $p_1, p_2, \ldots, p_n$ is equal to their product $p_1 p_2 \ldots p_n$.

Theorem 4.9. If ts.k.ch. z of mutual is simple with each ts.k.ch. $p_1, p_2, \ldots, p_n$, the z is mutually simple with product $p_1 p_2 \ldots p_n$.

Proof. According to condition $(z, p_k)=1\,(1\leqslant k\leqslant n)$, i.e. z and $p_k\,(1\leqslant k\leqslant n)$ are primes; therefore z and $p_1p_2\ldots p_n$ are primes, i.e., $(z, p_1p_2\ldots p_n)=1$.

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Chapter 2.

COMPARISONS OF COMPLEX INTEGERS AND THEIR PROPERTIES.

§1. Concept about the comparison.

In the previous chapter was determined the algorithm of the division in accordance with which any pair ts.k.ch. z and p can be compared with the pair ts.k.ch. q and r such, that

$$z = pq + r, \qquad (1,1)$$

where $|r| < |p|$.

Two ts.k.ch. z and w are called comparable in complex modulus p, if with division of z and w by p to them correspond one and the same remainder/residue; in this case they write:

$$z \equiv w \pmod{p}.$$

As has already been noted above, condition (1.4.1), generally speaking, does not provide single-valued determination the partial quotient q and residue r. Therefore it can seem that to one and the

same ts.k.ch. z can correspond several remainders/residues r_1, r_2 such, that $\|r_1\|<\|p\|$, $\|r_2\|<\|p\|$. But in that case, obviously,

$$r_1-r_2\equiv 0 \pmod{p}.$$

Actually/really,

$$z=q_1p+r_1,$$
$$z=q_2p+r_2,$$

where $\|r_1\|<\|p\|$, $\|r_2\|<\|p\|$.

Hence $r_1-r_2=(q_2-q_1)p$, i.e., $r_1-r_2=0 \pmod{p}$.

Example. $p=3+4i$, $\|p\|=25$.

$$5+6i=1\cdot(3+4i)+(2+2i),\ q_1=1,\ r_1=2+2i,\ \|r_1\|=8,$$
$$5+6i=2(3+4i)+(-1-2i),\ q_2=2,\ r_2=-1-2i,\ \|r_2\|=5,$$

but however, in this case $r_1-r_2=(2+2i)-(-1-2i)=3+4i\equiv 0 \pmod{p}$.

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§2. Basic properties of comparisons of ts.k.ch.

Property 1. If $z\equiv w$, $w\equiv u \pmod{p}$, то $z\equiv u \pmod{p}$.

Proof. Since $z-w=m_1p$, $w-u=m_2p$, the $w=u+m_2p$, therefore, $z-u=(m_1-m_2)p$.

Property 2. If

$$z_1 \equiv r_1 \pmod p,$$
$$z_2 \equiv r_2 \pmod p,$$

then

$$z_1 \pm z_2 \equiv r_1 \pm r_2 \pmod p,$$
$$z_1 \cdot z_2 \equiv r_1 \cdot r_2 \pmod p.$$

It is possible, in particular, to count $\|r_1\| < \|p\|$, $\|r_2\| < \|p\|$.

Proof. Since $z_1 = q_1 p + r_1$, $z_2 = q_2 p + r_2$, then

$$z_1 \pm z_2 = (q_1 \pm q_2) \cdot p + (r_1 \pm r_2),$$
$$z_1 \cdot z_2 = (q_1 q_2 p + q_1 r_2 + q_2 r_1)\, p + r_1 r_2.$$

The corollaries of this property are the following:

a) if $z + w \equiv u \pmod p$, then $z \equiv u - w \pmod p$;

b) if $z \equiv w \pmod p$, then for any ts.v.ch. u

$$z + u \equiv w + u \pmod p;$$

c) if $z \equiv w \pmod p$ and n - ts.v.ch. (positive), then $z^n \equiv w^n \pmod p$;

d) if $z \equiv w \pmod p$ and $v \in \Gamma$ then $vz \equiv vw \pmod p$;

e) if $A_k \equiv B_k \pmod p$ and $z \equiv w \pmod p$, then $\sum_{k=1}^{n} A_k z^k \equiv \sum_{k=1}^{n} B_k w^k \pmod p$.

Property 3. If $zq \equiv wq \pmod p$, moreover $(q, p) = 1$, then

Proof. According to the condition we have

$$(z-w)q=m_1p,$$

since q/m_1p and $(q, p)=1$, then q/m, therefore,

$$z-w\equiv 0 \pmod p.$$

Property 4. If $z\equiv w \pmod{p_i}$ $1\leqslant i<n$, moreover $p_1, p_2, \ldots, p_n$ pair-wise mutually simple and $P=p_1p_2\ldots p_n$, then $z\equiv w \pmod P$.

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Proof. According to theorem 1.4.8, $P/z-w$.

Property 5. If $z\equiv w (mod\, p)$ and $p=qd$, $\|q\|\neq 1$, then $z\equiv w \pmod q$.

Proof. Since $z-w=mp$ and $p=qd$, then $z-w=mdq$, i.e. $z-w\equiv 0 \pmod q$.

§3. Concept about the full/total/complete system of deductions.

On the given one $\bmod p\ (p\in\Gamma)$ the set of all ts.k.ch. is divided/marked off into the set of the nonintersecting sets each of which joins in itself all ts.k.ch. congruent between themselves in mod p.

A and B - two such sets. Let us show that they do not intersect. For this purpose let us take arbitrary elements/cells $z\in A$ and $w\in B$.

It is clear that $w \not\equiv z \pmod{p}$, since if would prove to be, that $w \equiv z \pmod{p}$, then in view of property 1 of comparisons arbitrary ts.k.ch. from A would prove to be comparable with arbitrary ts.k.ch. from B and thereby regarding both set A and B they would prove to be coincident.

This separation of set Γ into the subsets is called separation into the classes of equivalences (in this case two ts.k.ch. are considered equivalent, if they are congruent between themselves in mod p).

Each element/cell of the class of equivalency is called by the representative of class. It is obvious, any representative of class uniquely determines entire class.

The set of all classes of equivalency is called a factor-set of ring Γ on element p and is designated by symbol $\Gamma/(p)$. Using arithmetic properties of comparisons (property 2), it is not difficult to show that $\Gamma/(p)$ is ring. With this if z and w are the representatives of classes A_z and A_w, i.e., $z \to A_z$ and $w \to A_w$, then

$$\begin{aligned} z+w \to A_z + A_w = A_{z+w}, \\ z \cdot w \to A_z \cdot A_w = A_{z \cdot w}. \end{aligned} \tag{3.1}$$

The classes of equivalency on mod p are called also residue classes on mod p.

If we from e .h residue class select somehow or one and only for one representative (deduction), then we will obtain the full/total/complete system of the deductions (abbreviated as p.s.v.), which is characterized by the fact that

a) all numbers of this system are not congruent between themselves in mod p,

b) any ts.k.ch. of Γ is congruent with one of the numbers of this system.

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In accordance with the determination of p.s.v. is valid the following theorem.

Theorem 3.1. Set σ of ts.k.ch. forms p.s.v. on modulus/module p when and only when any ts.k.ch. $z \in \Gamma$ is unambiguously represented in the form

$$z = qp + r, \qquad (3.3)$$

where

$$r \in \sigma. \qquad (3.4)$$

Proof. Let with the preset set of numbers σ any ts.k.ch. in a single manner is represented in the form

$$z = qp + r,$$

where $r \in \sigma$.

Let us show that in that case the set σ forms p.s.v. on mod p. Actually/really, $r_1, r_2 \in \sigma$ and $r_1 \neq r_2$. Let us show that $r_1 \not\equiv r_2 \pmod p$.

Let us assume that $r_1 \equiv r_2 \pmod p$, then $r_1 = r_2 + m \cdot p$ $(m \neq 0)$ and, therefore, ts.k.ch. r_1 in the form

$$r_1 = r_1 + 0p,\ r_1 \in \sigma,$$
$$r_1 = r_2 + mp,\ r_2 \in \sigma,$$

which represented contradicts the uniqueness of the disintegration of any ts.k.ch. into form of form (3.3, 3.4).

Thus, from condition $r_1, r_2 \in \sigma$ and $r_1 \neq r_2$ it follows that $r_1 \not\equiv r_2$ (mod p). it remains to be convinced of the completeness of set σ. Completeness σ follows from the fact that any ts.k.ch. is represented in the form (3.3, 3.4). Reverse assertion is the immediate consequence of the determination of p.s.v.

Theorem 3.2. If σ_1 and σ_2 - two different p.s.v. in one and the same composite modulus/module p, then sets σ_1 and σ_2 are isomorphic, in other words, between the sets σ_1 and σ_2 there is one-to-one

conformity such, which from the condition

follows

$$\alpha_1 \longleftrightarrow \beta_1, \quad \alpha_1, \alpha_2 \in \sigma_1,$$
$$\alpha_2 \longleftrightarrow \beta_2, \quad \beta_1, \beta_2 \in \sigma_2$$
$$\alpha_1 + \alpha_2 \longleftrightarrow \beta_1 + \beta_2,$$
$$\alpha_1 \cdot \alpha_2 \longleftrightarrow \beta_1 \cdot \beta_2.$$

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Proof. According to theorem 3.1 any element/cell $\alpha \in \sigma_1$ is unambiguously represented in the form

$$\alpha = \beta + kp,$$

where

$$\beta \in \sigma_2,$$

moreover to two different elements/cells $\alpha_1, \alpha_2 \in \sigma_1$ in this way are compared two different elements/cells $\beta_1, \beta_2 \in \sigma_2$:

$$\alpha_1 = \beta_1 + k_1 p,$$
$$\alpha_2 = \beta_2 + k_2 p$$

Actually/really, from the assumption $\beta_1 = \beta_2$ it would follow that $\alpha_1 \equiv \alpha_2 \pmod{p}$. The latter is impossible, since $\alpha_1 \neq \alpha_2$ and $\alpha_1, \alpha_2 \in \sigma_1$. The isomorphism of conformity follows from the fact that

$$\alpha_1 + \alpha_2 = (\beta_1 + \beta_2) + (k_1 + k_2) p,$$
$$\alpha_1 \cdot \alpha_2 = \beta_1 \beta_2 + (\beta_1 k_2 + \beta_2 k_1 + k_1 k_2 p) p_1.$$

That presented above can be summarized as follows: the operation of comparison of ts.k.ch. on the preset modulus/module p realizes the single-valued (but not one-to-one) representation of ring Γ onto

quotient ring $\Gamma/(p)$, in this case circular operations (addition, multiplication) on the elements/cells on factor-ring $\Gamma/(p)$ are equivalent circular by the operations of ring Γ on mod p above the representatives of each class of equivalency. In other words, the study of the properties of comparisons is equivalent to the study of the properties of the operations of quotient ring $\Gamma/(p)$.

Let us illustrate this position on some important properties of the theory of comparisons.

Theorem 3.3. If ts.k.ch. w satisfies condition (w, p)=1 and if z passes p.s.v. on mod p, then ts.k.ch. wz+ζ, where ζ - any ts.k.ch., also it passes p.s.v. on mod p.

Proof. Let us designate through n(p) a number of elements/cells of p.s.v. on mod p. It will be shown below that for any ts.k.ch. p $n(p)=\|p\|$.

Since z accepts n(p) of different values, then the linear form wz+ζ also assume n(p) values. Let us show that if $z_1 \not\equiv z_2 \pmod p$, then

$$wz_1+\zeta \not\equiv wz_2+\zeta \pmod p.$$

Let us assume that

$$wz_1+\zeta \equiv wz_2+\zeta \pmod p,$$

then

$$wz_1 \equiv wz_2 \pmod p,$$

since $(w,p)=1$, then hence it follows that

$$z_1 \equiv z_2 \pmod{p},$$

which contradicts initial condition.

Let us pause at some corollaries of this.

Below we will assume that p.s.v. on mod p includes numbers 0 and 1 belong to the different classes of equivalency. Let us further note that p.s.v. which is formed by the values of the linear form of $wz+\zeta$ can be compared with p.s.v. whose value accepts the variable/alternating z. Then theorem 3.3 can be paraphrased as follows.

Theorem 3.4. If $(w,p)=1$, then the comparison

$$wz + \zeta \equiv u \pmod{p}, \qquad (3.5)$$

where ζ and u - any ts.k.ch., has single solution, belonging to p.s.v.

Observation. Let z_0 - solution of comparison (3.5) in the sense indicated, then, obviously, any ts.k.ch. from the class of equivalency, representative of which is z_0 it is solution (3.5).

Theorem 3.5. If $(w, p)=1$, then the comparison

$$wz \equiv 1 \pmod{p} \qquad (3.6)$$

has singularly solution of w^{-1}, belonging to p.s.v.

Example. To find the solution of the comparison

$$(1+2i)z \equiv 1 (\mathrm{mod}\ 4+i).$$

Here

$$(1+2i,\ 4+i)=1,$$

therefore comparison has the unique solution, which belongs to p.s.v., i.e.,

$$z \equiv w^{-1} = \frac{1}{1+2i} \equiv -1+i\ (\mathrm{mod}\,(4+i)).$$

Theorem 3.5 is corollary 3.3. In this case the residue of ts.k.ch. w^{-1} on mod p is called reverse to deduction ts.k.ch. w.

Corollary. If p - p.ch. Γ., then condition $(w, p)=1$ is satisfied for any $w \in \Gamma$. Consequently, if p - p.ch. Γ., then any element/cell from p.s.v. has reverse, i.e., p.s.v. on the simple modulus/module is formed field.

In the language of quotient ring $\Gamma/(p)$ the properties indicated can be formulated as follows.

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1. Element/call $A_w \in \Gamma/(p)$, whose all representatives mutually simple with p, it is divider/denominator of unit $\Gamma/(p)$; as unit of quotient ring $\Gamma/(p)$ it is accepted class of equivalencies which contains in itself 1.

2. Quotient ring $\Gamma/(p)$ is field, if p-p. k. Γ.;

3. If A_w - divider/denominator of unity of quotient ring $\Gamma/(p)$ and A_z passes all elements/cells $\Gamma/(p)$, then linear form

$$A_w \cdot A_z + A'_t$$

takes all values from $\Gamma/(p)$; in other words, if A_w - divider/denominator of unity from $\Gamma/(p)$, then equation

$$A_w \cdot A_z + A'_t = A_n$$

has unique solution with any A_n, A'_t.

Thus, symbol of comparison (≡) in the language of quotient ring $\Gamma/(p)$ or, which the same, of cell of elements/cells, p.s.v. acquires the sense of the symbol of equality (=).

Since arithmetic ts.k.ch. in different numeration systems deals concerning the elements/cells of p.s.v., then it is expedient from the symbol of comparison to pass to the symbol of equality. For this purpose the chosen set of p.s.v. on mod p we will designate by the symbol

$$<\cdot|_p .$$

The operation of extraction from number z on the basis of modes p of residue r, belonging to that chosen p.s.v. $<\cdot|_p$, we will designate by the symbol

$$<z|_p.$$

In other words, if z=qp+r and $r\in<\cdot|_p$, then $r=<z|_p$.

In the language of these symbols the mentioned agreements and operations will take the following form: 1) $0,1\in<\cdot|_p$ for any p, $\|p\|\neq1$; 2) $<z|_p$ — ts.v.ch. 3) $\ll z|_p|_p=<z|_p$; 4) $<z+w|_p=\ll z|_p+<w|_p|_p$; 5) if $<z+w|_p=<u|_p$, then $<z|_p=\ll u|_p-<w|_p|_p$; 6) $<zw|_p=\ll z|_p\cdot<w|_p|_p$; 7) if $(z, p)=1$, then there exists the single remainder $<z^{-1}|_p$ such that $\ll z|_p<z^{-1}|_p|_p=1$.

Observations. 1. In ring Γ symbol $<z|_p$ is analog of symbol $|x|_p$ in ring of ts.v.ch. [11], i.e., deduction of integer or real modulus/module p.

2. Since for complex numbers the concepts "more" and "less" are deprived of sense, the participation of triangular parenthesis in recording $<z|_p$ cannot introduce ambiguity.

For the image of elements/cells of p.s.v. henceforth we will use mainly Greek letters α, β, γ,....

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4. For image of p.s.v. on real modulus/module we will use symbol $|\cdot|_p$.

§4. Primitive roots and indices.

Theorem 4.1. If ts.k.ch. w satisfies condition $p \nmid w$ and $\|p\| = N$, then

$$w^{N-1} \equiv 1 \pmod p.$$

Proof. Let σ indicate set of full/total/complete system of deductions $\langle\cdot\rangle_p = \{\xi, \eta, \varepsilon, \ldots\}$, from which is excluded the deduction, which separates into p.

We form the set of the products

$$\sigma' = \{w\xi, w\eta, w\varepsilon, \ldots\}.$$

It is obvious.

$$w\xi \equiv \xi' \pmod p, \quad w\eta \equiv \eta' \pmod p,$$

$$w\varepsilon \equiv \varepsilon' \pmod p, \ldots, \text{ где } \xi', \eta', \varepsilon', \ldots \in \sigma, \tag{4.1}$$

since according to theorem condition none of the numbers of sets σ' is divided into p. The set of numbers $\xi', \eta', \varepsilon', \ldots$ let us designate through σ''. We now form the products of single numbers of sets σ, σ', σ'', taking into account that a quantity of numbers in each set is equal N-1:

$$M = \xi\eta\varepsilon\ldots,$$
$$M' = w^{N-1}\xi\eta\varepsilon\ldots = w^{N-1}M,$$
$$M'' = \xi'\eta'\varepsilon'\ldots.$$

Then by force (4.1)

$$M'' \equiv M' \pmod{p},$$

or

$$M'' \equiv w^{N-1}M \pmod{p}, \qquad (4.2)$$

but

$$M'' = M,$$

since numbers $\xi', \eta', \varepsilon', \ldots$ of set σ'' compulsorily coincide with numbers of set σ in view of condition $\sigma'' \in <\cdot|_p$.

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Therefore from (4.2) we have

$$(w^{N-1} - 1)M \equiv 0 \pmod{p},$$
$$p \mid (w^{N-1} - 1)M,$$

whence, on the strength of the fact that $p \times M$, we have $p \mid (w^{N-1} - 1)$, which is equivalent to

$$w^{N-1} \equiv 1 \pmod{p}.$$

Theorem 4.2. If $p \times w$, and t while t is the smallest index for which $w^t \equiv 1 \pmod{p}$, then t is the divider/denominator of any other index k,

for which

$$w^k \equiv 1 \pmod p. \qquad (4.3)$$

Proof. Let $t \times k$, then is found such ts.v.ch. n for which difference nt-k<t. In view of the property of comparisons from theorem condition we have

$$w^{nt} \equiv 1 \pmod p. \qquad (4.4)$$

Further from (4.3) and (4.4) we get

$$w^{nt} - w^k = w^k (w^{nt-k} - 1) = 0 \pmod p,$$

whence, since we have $p \not\mid w^k$,

$$w^{nt-k} \equiv 1 \pmod p.$$

Consequently, there is a degree of number w with the index less than t, for which is satisfied the condition $w^t \equiv 1 \pmod p$, and this contradicts theorem condition.

Corollary $t \mid N-1$ $(N = \| p \|)$ or t=N-1, since according to theorem 4.1 $w^{N-1} \equiv 1 \pmod p$.

If t=N-1, then w is called primitive roots on mod p. The following theorem shows that this primitive roots exists.

Theorem 4.3. Let N-1 have the disintegration into the simple multipliers

$$N - 1 = a^\alpha b^\beta c^\gamma, \ldots, (N = \| p \|), \qquad (4.5)$$

where a, b, c, ... - prime numbers, and let it be further A, B, B,

... - ts.k.ch. not divisible into p numbers which do not satisfy with respect to the comparisons

$$x^{\frac{N-1}{a}} \equiv 1 \pmod{p},$$

$$x^{\frac{N-1}{b}} \equiv 1 \pmod{p},$$

$$x^{\frac{N-1}{c}} \equiv 1 \pmod{p},$$

$$\dots\dots\dots\dots$$

then, if ts.k.ch. w satisfies the condition

$$A^{\frac{N-1}{a^\alpha}} \cdot B^{\frac{N-1}{b^\beta}} \cdot C^{\frac{N-1}{c^\gamma}} \dots \equiv w \pmod{p},$$

w is primary root on mod p.

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Proof. Let us assume that w is not primitive roots. This means that index t - smallest congruent with unity degree of number w is divider/denominator N-1 (see theorem 4.2). Consequently, $\frac{N-1}{t}$ is an integer, greater than unity. It is obvious, the simple multipliers of a number are contained among number a, b, c, ...

Let a be the divider/denominator of this number, i.e., N-1=t r a, but by hypothesis $w^t \equiv 1 \pmod{p}$, therefore $w^{tr} \equiv 1 \pmod{p}$, either $w^{\frac{N-1}{a}} \equiv 1 \pmod{p}$, or

$$A^{\frac{N-1}{a^\alpha}\frac{N-1}{a}} B^{\frac{N-1}{a^\beta}\frac{N-1}{a}} C^{\frac{N-1}{c^\gamma}\frac{N-1}{a}} \dots \equiv 1 \pmod{p}. \qquad (4.6)$$

From (4.5) it is evident that $\frac{N-1}{ab^\beta}$ is ts.v.ch. and therefore in vi...

of theorem 4.1 it follows that

$$B^{\frac{N-1}{b^3}\cdot\frac{N-1}{a}}=(B^{N-1})^{\frac{N-1}{ab^3}}\equiv 1 \pmod p$$

and in exactly the same manner $\frac{N-1}{ac^{\gamma}}$ is ts.v.ch.; therefore

$$C^{\frac{N-1}{c^{\gamma}}\cdot\frac{N-1}{c}}\equiv 1 \pmod p.$$

Consequently, from (4.6) we consist that

$$A^{\frac{N-1}{a^2}\cdot\frac{N-1}{a}}\equiv 1 \pmod p. \qquad (4.7)$$

Further we select ts.v.ch. $\lambda>0$ such, that $\lambda b^3c^{\gamma}\ldots\equiv 1 \pmod a$. This is possible, because $(b^3c^{\gamma}\ldots, a)=1$.

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Let us assume now $\lambda b^3c^{\gamma}\ldots=1+a\mu$, then, obviously, from (4.7) we have

$$A^{\lambda\frac{N-1}{a^2}\cdot\frac{N-1}{a}}\equiv 1 \pmod p$$

or since

$$\lambda\frac{N-1}{a^2}\cdot\frac{N-1}{a}=(1+a\mu)\frac{N-1}{a}=(N-1)\mu+\frac{N-1}{a},$$

that

$$A^{\lambda\frac{N-1}{a^2}\cdot\frac{N-1}{a}}=A^{(N-1)\mu+\frac{N-1}{a}}\equiv 1 \pmod p$$

or

$$A^{(N-1)\mu}A^{\frac{N-1}{a}}\equiv 1 \pmod p,$$

but according to theorem 4.1 $A^{(N-1)\mu}\equiv 1 \pmod p$, therefore probably

$$A^{\frac{N-1}{2}} \equiv 1 \pmod p,$$

but this contradicts theorem condition.

Theorem 4.4. If w is primitive roots on mod of p, then the terms of the series/row

$$1,\ w,\ w^2,\ w^3, \ldots,\ w^{N-2} \qquad (4.8)$$

comprise the total system of deductions on mod p, with exception of zero deduction.

Proof. Let us suppose series/row (4.8) does not comprise $\langle\cdot|_p$, then among the members of this series/row are located such members, as satisfy the condition

$$w^l \equiv w^k \pmod p,$$

where for the clarity let us assume 0≤k<l<N-1 or

$$w^{l-k} \equiv 1 \pmod p,$$

but this contradicts the determination of primitive roots.

And here, as in the case of real numbers, it is possible to introduce the concept of index.

If $A \equiv w^{\mu} \pmod p$ (but this always occurs according to theorem 4.4), where A - ts.k.ch., then μ is called index A on mod p with basis/base p with base w and is designated by the symbol

$$\mu = \operatorname{ind}_w A.$$

It is obvious, has the place

$$\operatorname{ind} ABC \ldots KL \equiv \operatorname{ind} A + \operatorname{ind} B + \operatorname{ind} C + \ldots + \\ + \operatorname{ind} K + \operatorname{ind} L.$$

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Actually/really, we have

$$A \equiv w^{\operatorname{ind} A} (\operatorname{mod} p),$$
$$B \equiv w^{\operatorname{ind} B} (\operatorname{mod} p),$$
$$\cdots\cdots\cdots$$
$$L \equiv w^{\operatorname{ind} L} (\operatorname{mod} p)$$

and after multiplication we will obtain

$$AB \ldots L \equiv w^{\operatorname{ind} A + \operatorname{ind} B + \ldots + \operatorname{ind} L} (\operatorname{mod} p),$$

whence the sum

$$\operatorname{ind} A + \operatorname{ind} B + \ldots + \operatorname{ind} L,$$

being one of the indices of product A•B•C... •L, is equal to

$$\operatorname{ind} A \cdot B \cdot C \ldots L \equiv \operatorname{ind} A + \operatorname{ind} B + \operatorname{ind} C + \ldots + \operatorname{ind} L.$$

Example. Let us look the table of indices for mod4+i. Here p=4+i, $N = |4+i| = 17$, and as the primitive roots can be taken w=-1-i, since

$$(-1-i)^{16} \equiv 1 \ (\operatorname{mod} 4+i).$$

Therefore we find

$$(-1-i)^0 \equiv 1. \qquad (-1-i)^8 \equiv -1.$$
$$(-1-i)^1 \equiv -1-i. \qquad (-1-i)^9 \equiv 1+i.$$
$$(-1-i)^2 \equiv 2i. \qquad (-1-i)^{10} \equiv -2i.$$
$$(-1-i)^3 \equiv 1-2i. \qquad (-1-i)^{11} \equiv -1-2i.$$
$$(-1-i)^4 \equiv i. \qquad (-1-i)^{12} \equiv -i.$$
$$(-1-i)^5 \equiv 1-i. \qquad (-1-i)^{13} \equiv -1+i.$$
$$(-1+i)^6 \equiv -2. \qquad (-1-i)^{14} \equiv 2.$$
$$(-1+i)^7 \equiv -2-i. \qquad (-1-i)^{15} \equiv 2-i.$$

and table will be represented in the following form:

(1) Индекс	(2) Вычет	(1) Индекс	(2) Вычет	(1) Индекс	(2) Вычет
0	1	6	-2	12	$-i$
1	$-1-i$	7	$-2+i$	13	$-1+i$
2	$2i$	8	-1	14	2
3	$1+2i$	9	$1-i$	15	$2-i$
4	i	10	$-2i$		
5	$1-i$	11	$-1-2i$		

Key: (1). Index. (2). Deduction.

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Chapter 3.

MOST IMPORTANT METHODS OF THE ASSIGNMENT OF THE COMPLETE SYSTEMS OF RESIDUES. ANALYSIS OF TABLES OF MODULAR ARITHMETIC.

§1. Method of the assignment of the full/total/complete system of deductions.

Let p=c+di - modulus/module and z=a+bi - arbitrary ts.k.ch. We have

$$\frac{a+bi}{c+di}=\frac{(a+bi)(c-di)}{\|p\|}=\frac{ac+bd}{\|p\|}+i\frac{bc-ad}{\|p\|}. \qquad (1.1)$$

Let us assign certain of p.s.v. $|\cdot|_{\|p\|}$ on real modulus/module $\|p\|$. Then the following expansions are determined by the single form:

$$\begin{aligned} ac+bd &= m_1\|p\|+|ac+bd|_{\|p\|},\\ bc-ad &= m_2\|p\|+|bc-ad|_{\|p\|}. \end{aligned} \qquad (1.2)$$

Here, as it was stipulated earlier, by symbol $|x|_{\|p\|}$ is designated deduction of ts.v.ch. X on real modulus/module $\|p\|$.

In accordance with equalities (1.1) and (1.2) we will obtain

$$a+bi=(m_1+im_2)p+(\,|ac+bd|_{|p|}+$$
$$+i\,|bc-ad|_{|p|})\frac{p}{|p|}. \qquad (1.3)$$

Since a+bi, m_1+m_2i, c+di - ts.k.ch., then value

$$(\,|ac+bd|_{|p|}+i\,|bc-ad|_{|p|})\frac{p}{|p|} \qquad (1.4)$$

is ts.k.ch.

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Since expansions (1.1) and (1.2) are determined in a single manner, expansion (1.3) is also determined in a single manner, i.e., for any ts.k.ch. a+bi formula (1.4) in a single manner determines remainder/residue from the division of number a+bi into c+di. Is consequently, in view of theorem 2.3.1, valid the formula

$$\langle a+bi\,|_p=(\,|ac+bd|_{|p|}+i\,|bc-ad|_{|p|})\frac{p}{|p|}, \quad (1.5)$$

from which it follows that the method of assignment of p.s.v. with respect to the composite modulus/module p=c+di depends on the method of assignment of p.s.v. with respect to real modulus/module $\|p\|$.

Formula (1.5) we will call the basic formula of deductions on the composite modulus/module p=c+di.

From basic formula (1.5) follows the identity for the

elements/cells of p.s.v. If $\alpha-i\beta\in<\cdot|_p$, then

$$\alpha+i\beta=(|\alpha c+\beta d|_{\|p\|}+i|\beta c-\alpha d|_{\|p\|})\frac{p}{\|p\|}. \quad (1.6)$$

Actually/really, on one hand, on the strength of the fact that $\alpha+i\beta\in<\cdot|_p$, we have $<\alpha+\beta i|_p=\alpha-\beta i$. On the other hand, by force (1.5), we have

$$<\alpha+\beta i|_p=(|\alpha c+\beta d|_{\|p\|}+i|\beta c-\alpha d|_{\|p\|})\frac{p}{\|p\|}.$$

Consequently, relationship/ratio (1.6) is correct for any ts.k.ch.

$$\alpha+i\beta\in<\cdot|_p.$$

Hence, in particular, we will obtain that if $\alpha+i\beta\in<\cdot|_p$, then

$$|\alpha c+\beta d|_{\|p\|}=\alpha c+\beta d,$$
$$|\beta c-\alpha d|_{\|p\|}=\beta c-\alpha d. \quad (1.7)$$

Let us consider a question about quantity n(p) of elements/cells of p.s.v. concerning the composite modulus/module p.

Correctly following assertion.

Theorem 1.1. A number of elements/cells of p.s.v. on the composite modulus/module is equal to the norm of modulus/module p, i.e.

$$n(p)=\|p\|.$$

Proof. In accordance with formulas (1.7) the set of ts.k.ch. of form x_1+ix_2 is formed p.s.v. on mod p when and only when x_1 and x_2 are the integral solutions of the systems of the form

$$cx_1+dx_2=y_1,$$
$$-dx_1+cx_2=y_2, \quad (1.8)$$

where y_1 and y_2 pass the values of certain p.s.v. on real mod $\|p\|$.

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Consequently, a question about a number of elements/cells of p.s.v. concerning the composite modulus/module p is reduced to a question about a number of pairs (y_1, y_2), for which systems (1.8) have the integral solutions.

Since from (1.8) it follows that

$$x_1 = \frac{cy_1 - dy_2}{\|p\|}, \qquad x_2 = \frac{cy_2 + dy_1}{\|p\|}, \tag{1.9}$$

then the stated requirement satisfy only those pairs (y_1, y_2), for which it is carried out

$$\begin{cases} cy_1 - dy_2 \equiv 0 \\ cy_2 + dy_1 \equiv 0 \end{cases} \pmod{\|p\|} \tag{1.10}$$

Subsequently it suffices to consider three cases.

Case 1. $c \neq 0$, $d = 0$, $(\|p\| = c^2)$. System (1.10) takes the form

$$\begin{cases} cy_1 \equiv 0 \\ cy_2 \equiv 0 \end{cases} \pmod{c^2}.$$

Each of these comparisons has s of different (incomparable on mod c) solutions and, therefore, in all different pairs (y_1, y_2), which satisfy system (1.10), there will be c^2, i.e., $\|p\|$.

Case of 2. c, $d \neq 0$, moreover (c, d) = 1. In this case appears the possibility of resolving one of comparisons (1.10) relative to any variable/alternating, since $(c, \|p\|)=1$ and $(d, \|p\|)=1$.

Solving, for example, the first comparison is relative y_1, we will obtain $y_1 = |c^{-1} dy_2|_{\|p\|}$.

Giving to value y_2 values from p.s.v. mod $\|p\|$, we will obtain $\|p\|$ different pairs of numbers (y_1, y_2), which satisfy the first comparison of system (1.10). Let us show that these pairs satisfy the second comparison.

Actually/really, we have

$$|cy_2 + dy_1|_{\|p\|} = |cy_2 + d\,|c^{-1} dy_2|_{\|p\|}|_{\|p\|} =$$
$$= |c^{-1}(c^2 + d^2)\, y_2|_{\|p\|} = 0.$$

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Thus, a number of different pairs (y_1, y_2), which satisfy system (1.10), is equal to $\|p\|$.

Case of 3. $p=\rho p_1$, where $p_1=c_1+d_1 i$, moreover $(c_1, d_1)=1$ and ρ - ts.v.ch. Systems (1.8) take the form

$$\begin{cases} \rho(c_1 x_1 + d_1 x_2) = y_1, \\ \rho(-d_1 x_1 + c_1 x_2) = y_2. \end{cases}$$

From this it follows that for the integrality of the solutions of system (1.8) in the class of pairs (y_1, y_2) it is necessary to examine those deductions y_1 and y_2 on mod$\|p\|$, which satisfy the requirements ρ/y_1; ρ/y_2.

Let $y_1=\rho r_1$, $y_2=\rho r_2$. In accordance with this system (1.10) will take the form

$$\begin{cases} \rho^2(c_1 r_1 - d_1 r_2) \equiv 0 \\ \rho^2(c_1 r_2 + d_1 r_1) \equiv 0 \end{cases} \pmod{\rho^2 \| p_1 \|}. \qquad (1.11)$$

The solutions of system (1.10) can be obtained as follows. Reducing system (1.11) on ρ^2, we will obtain

$$\begin{cases} c_1 r_1 - d_1 r_2 \equiv 0 \\ c_1 r_2 + d_1 r_1 \equiv 0 \end{cases} \pmod{\| p_1 \|}.$$

The general solution of this system is written/recorded in the form

$$r_1 = r_1^{(0)} + k_1 \| p_1 \|,$$

$$r_2 = r_2^{(0)} + k_2 \| p_1 \|,$$

where $r_2^{(0)} \in | \cdot |_{\|p_1\|}$ and $r_1^{(0)} = | c_1^{-1} d_1 r_2 |_{\|p_1\|}$.

In accordance with this the set of the solutions of system

(1.10) will take the form

$$y_1 = pr_1^{(0)} + pk_1 \| p_1 \|,$$
$$y_2 = pr_2^{(0)} + pk_2 \| p_1 \|.$$

Hence a quantity of different pairs (y_1, y_2), where $y_1, y_2 \in [0, \|p\|]$, that satisfy condition (1.10), will be equally to $\|p\|$, since a quantity of pairs $(r_1^{(0)}, r_2^{(0)})$ is equal to $\|p_1\|$, and values k_1 and k_2 can take independently only p different values (deductions on mod p).

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Observations. 1. As can be seen from proof, in the case of 2 requirement (c, d)=1 reduces dimension of system (1.10). It will be evident below that this requirement makes it possible to introduce one-dimensional p.s.v. on the composite moduli/modules, i.e., to examine p.s.v., on the composite moduli/modules those wholly consisting, for example, of the real deductions.

2. Given proof of theorem 1.1 contains algorithm of construction of p.s.v. on composite modulus/module p. Since this algorithm has independent interest, then it is expedient to formulate it independently.

Method of construction of p.s.v. on the composite modulus/module p=c+di.

Case 1. (d=0, c≠0).

P.s.v. is described by set ts.k.ch. x_1+ix_2, where x_1 and x_2 independently pass values of p.s.v. $|\cdot|_{c^2}$ on the the real mod c^2.

Case 2. (d≠0, c≠0, (c, d)=1).

P.s.v. is described by set ts.k.ch. x_1+ix_2:

$$\begin{aligned} x_1 &= \frac{1}{\|p\|}(c\,|\,c^{-1}dr\,|_{\|p\|}-dr), \\ x_2 &= \frac{1}{\|p\|}(cr+d\,|\,c^{-1}dr\,|_{\|p\|}), \end{aligned} \qquad (1.12)$$

where r passes value of p.s.v. $|\cdot|_{\|p\|}$ on real

Case 3. (c, d) $=\rho$, ($c=c_1\rho$, $d=d_1\rho$, $p=p_1\rho$).

P.s.v. is described by set ts.k.ch. x_1+ix_2:

$$x_1 = \frac{1}{\|p_1\|}(c_1\,|\,c_1^{-1}d_1r\,|_{\|p_1\|}-d_1r)+(c_1k_1-d_1k_2),$$

$$(1.13)$$

$$x_2 = \frac{1}{\|p_1\|}(c_1r+d_1\,|\,c_1^{-1}d_1r\,|_{\|p_1\|})+(c_1k_2+d_1k_1),$$

where r passes value of p.s.v. $|\cdot|_{\|p_1\|}$ on real mod $\|p_1\|$, and k_1 and k_2 accept independently ρ the values, incomparable on mod$\|p\|$.

§2. Most customary p.s.v. on the composite moduli/modules.

Generally speaking, the selection of one or the other system of deductions can be dictated by different considerations.

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As one of such considerations can serve, for example, the principle of tight packing, i.e., the requirement of that, so that ts.k.ch. that are the deductions on mod p, would tightly cover/coat certain geometric form.

Let us explain the character of the location of the integer points of composite plane, which represent ts.k.ch. divisible into modulus/module p.

Let p=c+di - preset modulus/module (c, d≠0).

We have

$$\frac{x+iy}{c+id}=k_1+ik_2 \quad k_1 \overset{(1)}{\text{и}} k_2 - \text{ц.} \overset{(2)}{\text{ч.}} \qquad (2.1)$$

Key: (1). and. (2). ts.ch.

Hence

$$cx+dy=k_1 \| p \|, \qquad (2.2)$$

$$-dx+cy=k_2 \| p \|. \qquad (2.3)$$

The joint solutions of these equations with the arbitrary wholes k_1 and k_2 describe the set of all ts.k.ch. separated into p. Straight lines (2.2) and (2.3) form orthogonal family. After leading the equations of these straight lines to the normal mode

$$\frac{c}{\operatorname{sign}k_1\sqrt{a^2+b^2}}x+$$
$$+\frac{d}{\operatorname{sign}k_1\sqrt{a^2+b^2}}y=$$
$$=|k_1|\sqrt{a^2+b^2},$$
$$\frac{-d}{\operatorname{sign}k_2\sqrt{a^2+b^2}}x+$$
$$+\frac{c}{\operatorname{sign}k_2\sqrt{a^2+b^2}}y=$$
$$=|k_2|\sqrt{a^2+b^2},$$

we consist that the parallel lines this family are arranged $\sqrt{a^2+b^2}$ equivalent at a distance from each other (Fig. 1).

The vector sense of the points, which represent ts.k.ch. that separate into p, is such. Let us present (2.1) in the form

$$x+iy=k_1(c+id)+k_2(-d+ci).$$

Hence it follows that the unknown points are the apexes/vertexes of the integral linear combination of the orthogonal vectors c+id and -d+ci.

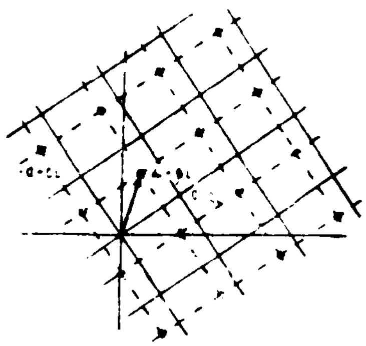

Fig. 1. Geometric representation of p.s.v. according to mod c+id (-) - straight lines (2.2) and (2.3); (*) - congruency points.

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In other words, if we initial integral grid consider as the integral coordinate system with the base vectors {1, i}, then the points, which represent ts.k.ch. that separate into p=c+id, represent the integral coordinate system with the orthogonal base {c+id, -d+ci}, obtained by the rotation of base (1, i) to the angle, equal to arg(c+id) (see Fig. 1). All integer points, which do not coincide with the apexes/vertexes of the squares examined, represent complex integers, which do not separate entirely into p=c+id.

Let us explain the new geometric location of all points, which represent congruent between themselves in mod p ts.k.ch., i.e., the

set of all points, which are determining the class of equivalency.

We have

$$x + iy = (k_1 + ik_2)p + \alpha + i\beta.$$

Hence we obtain

$$c(x-\alpha) + d(y-\beta) = k_1 |p|,$$
$$-d(x-\alpha) + c(y-\beta) = k_2 |p|.$$

being congruent/equating the obtained system of equations with system (2.2), (2.3), we consist that the unknown points are the apexes/vertexes of the squares of grid with the base {c+id, -d+ci}, in parallel to that moved the vector $\alpha+i\beta$. Such points we will call congruent (see Fig. 1).

In conclusion let us refine: in what cases and in what quantity on the sides of the congruent squares of grid with the base {c+id, -d+ci} can be placed the integer points?

Theorem 2.1. Let p=c+di - ts.k.ch.

If (c, d)=1, then on the sides of congruent squares are absent the integer points, different from apexes of squares; if (c, d) =ρ ($\rho \neq 1$), then a quantity of integer points on the sides of the congruent squares, different from the apexes/vertexes, is equal ρ-1.

Proof.

Case (c, d) =1. (x_0, y_0), (x_1, y_1) - two arbitrary integer points, which lie on the straight line (2.2). Let us show that the distance between them is not less $\sqrt{a^2+b^2}$.

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Since

$$cx_0 + dy_0 = k_1 \| p \|,$$
$$cx_1 + dy_1 = k_1 \| p \|,$$
$$c(x_1 - x_0) = d(y_1 - y_0). \qquad (2.4)$$

Since (c, d) =1, then

$$\begin{aligned} y_1 - y_0 &= \lambda c \\ x_1 - x_0 &= \mu d \end{aligned} \quad (\lambda,\ \mu - \text{ц. ч}).$$

Taking into account latter/last equalities, formula (2.4) let us rewrite in the form $cd\mu=cd\lambda$ or $\lambda=\mu$. Thus,

$$\sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2} = |\lambda| \sqrt{c^2 + d^2} \geqslant \sqrt{c^2 + d^2}.$$

Case (c, d) $=\rho$ $(\rho \neq 1)$. Let $c=c_1\rho$, $d = d_1\rho$, $\| p \| = \rho^2 \| p_1 \|$, then equations (2.2) and (2.3) take the form

$$\begin{aligned} c_1 x + d_1 y &= k_1 \rho \| p_1 \| \\ -d_1 x + c_1 y &= k_2 \rho \| p_1 \| \end{aligned} \quad (c_1, d_1) = 1.$$

After leading these equations to the normal mode, we are convinced, that the corresponding integral grid has a step/pitch

$\sqrt{c_1^2+d_1^2}$. But cuttings off with a length of $\sqrt{c_1^2+d_1^2}$ is placed exactly ρ once in segment $\sqrt{c^2+d^2}$.

Example 1. Figure 2 shows congruent squares the modulus/module 2+3i. On the sides of squares is arranged/located not one whole points, since (2.3)=1.

Example 2. For p=3+6i in figure 3 are constructed congruent squares. On each side of square they are arranged/located on 2 integer points, since $(3.6)=\rho=3$.

Observation. That presented makes it possible to give the following geometric interpretation of further components/terms/addends of formulas (1.13). For this purpose we convert formulas (1.13) to the form

$$x_1+ix_2=x_1^{(0)}+ix_2^{(0)}+k_1(c_1+d_1i)+k_2(-d_1-c_1i), \quad (1.13)'$$

where

$$\left.\begin{aligned} x_1^{(0)} &= \frac{1}{\|p_1\|}\left(c_1\,|c_1^{-1}d_1r|_{\|p_1\|}-d_1r\right) \\ x_2^{(0)} &= \frac{1}{\|p_1\|}\left(c_1r+d_1\,|c_1^{-1}d_1r|_{\|p_1\|}\right) \end{aligned}\right\} \quad r\in|\cdot|_{\|p_1\|}.$$

Hence it follows that the deductions on the composite/compound modulus/module mod $\rho(c_1+d_1i)$ are obtained by addition to each deduction on mod (c_1+d_1i) composite of the deductions on mod ρ, represented to scale of the integral coordinate system with the orthogonal base $\{c_1+d_1i,\ -d_1+c_1i\}$.

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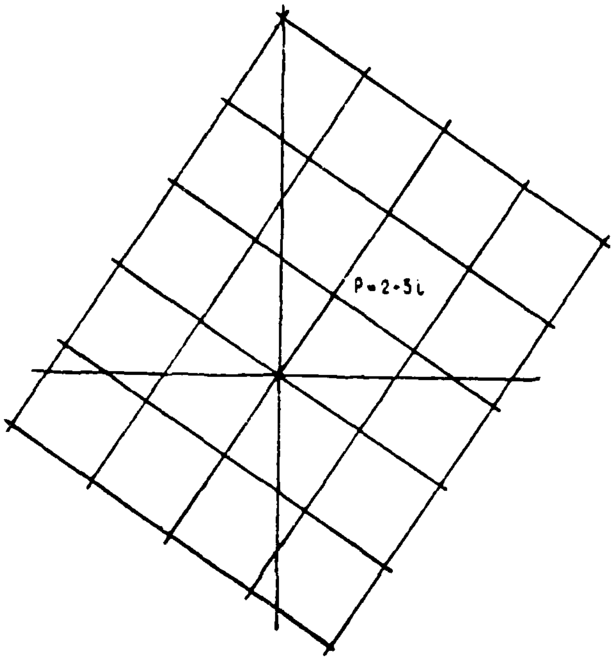

Fig. 2. Congruent squares on mod (2+3i).

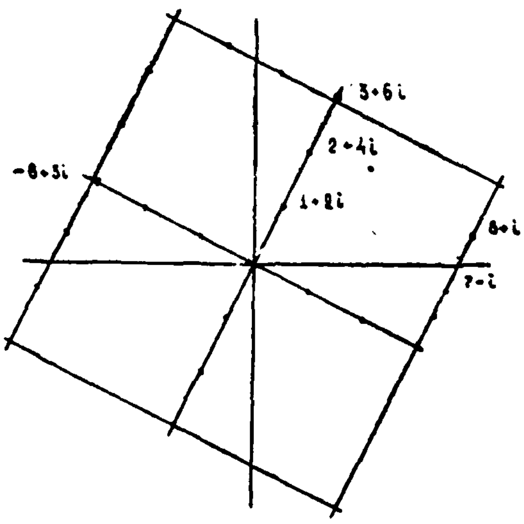

Fig. 3.

Fig. 3. Congruent squares on mod (3+6i).

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Let us now move on to the examination of some concrete/specific/actual types of p.s.v. on the composite moduli/modules. Let us preliminarily note that the full/total/complete systems of the smallest non-negative ones and the full/total/complete systems of the least positive residues on the real modulus/module we will designate respectively by the symbols:

$$|\cdot|_q^+ \text{— п. с. н.}^{(1)}\text{ н. в.}$$

$$|\cdot|_q^- \text{— п. с. а.}^{(2)}\text{ н. в.}$$

Key: (1). p.s.n.n.v. (2). p.s.a.n.v.

P.s.v. on the composite modulus/module p=c+di, determined by the formula

$$\langle a+bi\,|_p^+ = (\,|ac+bd\,|_{|p|}^+ + i\,|bc-ad\,|_{|p|}^+)\,\frac{p}{|p|},$$

is called the full/total/complete system of the smallest deductions on the composite modulus/module p (abbreviated as p.s.n.v.).

P.s.v. on the composite modulus/module p=c+di, determined by the formula

$$\langle a+bi\,|_p^- = (\,|ac+bd\,|_{|p|}^- + i\,|bc-ad\,|_{|p|}^-)\,\frac{p}{|p|},$$

is called the full/total/complete system of the least positive

residues on the composite modulus/module p (abbreviated as p.s.a.n.v.).

Let us give examples. For the calculation of p.s.n.v. and p.s.a.n.v. we will use the algorithm, described in the previous paragraph (theorem 1.1).

Example 1. mod 3 (case 1).

(1) п. с. н. в.			(2) п. с. а. н. в.		
0	1	2	$-1-i$	$-i$	$1-i$
i	$1+i$	$2+i$	-1	0	1
$2i$	$1+2i$	$2+2i$	$-1+i$	i	$1+i$

Key: (1). p.s.n.v. (2). p.s.a.n.v.

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Example 2. mod 3+2i (case of 2)

$(3, 2)=1,\ \|3+2i\|=13.$

a) P.s.n.v. $\langle\cdot|^{+}_{3+2i}=x_1+ix_2$, where

$$x_1=\frac{1}{13}(3\cdot|9\cdot 2r|^{+}_{13}-2r),\qquad x_2=\frac{1}{13}(3r+2\cdot|9\cdot 2r|^{+}_{13}),\qquad 0<r<12$$

r	$\|c^{-1}dr\|_{13}$	x_1	x_2	$<\cdot\|_{\overline{3+2i}}$
0	0	0	0	$0+0i$
1	5	1	1	$1+i$
2	10	2	2	$2+2i$
3	2	0	1	i
4	7	1	2	$1+2i$
5	12	2	3	$2+3i$
6	4	0	2	$2i$
7	9	1	3	$1+3i$
8	1	-1	2	$-1+2i$
9	6	0	3	$3i$
10	11	1	4	$1+4i$
11	3	-1	3	$-1+3i$
12	8	0	4	$4i$

b) P.s.a.n.v. $<\cdot|_{\overline{3+2i}} = x_1 + x_2 i$, where

$$x_1 = \frac{1}{13}(3\,|-4\cdot 2r\,|_{\overline{13}} - 2r),$$

$$-6 \leqslant r \leqslant 6$$

$$x_2 = \frac{1}{13}(3r+2\,|-4\cdot 2r\,|_{\overline{13}}).$$

r	$\|c^{-1}dr\|_{\overline{13}}$	x_1	x_2	$<\cdot\|_{\overline{3+2i}}$
-6	-4	0	-2	$-2i$
-5	1	1	-1	$1-i$
-4	6	2	0	2
-3	-2	0	-1	$-i$
-2	3	1	0	1
-1	-5	-1	-1	$-1-i$
0	0	0	0	0
1	5	1	1	$1+i$
2	-3	-1	0	-1
3	2	0	1	$+i$
4	-6	-2	0	-2
5	-1	-1	1	$-1+i$
6	4	0	2	$2i$

Example 3. mod $2+4i$, (case of 3).

$(2, 4)=2;\ c_1=1,\ d_1=2,\ \|p_1\|=5.$

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Let us calculate p.s.n.v. $<\cdot|^{-}_{2+4i}$ in accordance with (1.13):

$$x_1 + ix_2 = \left(\frac{1}{5}(|2r|^{+}_{5} - 2r) + \frac{i}{5}(r+2\cdot|2r|^{+}_{5})\right) +$$
$$+ k_1(1+2i) + k_2(-2+i),$$
$$r = 0, 1, 2, 3, 4; \quad k_1, k_2 = 0, 1.$$

Table of values $x_1^{(0)}, x_2^{(0)}$

r	$\|2r\|^{-}_{5}$	$x_1^{(0)}$	$x_2^{(0)}$	$<\cdot\|^{-}_{1-2i}$
0	0	0	0	$0+0i$
1	2	0	1	i
2	4	0	2	$2i$
3	1	-1	1	$-1-i$
4	3	-1	2	$-1+2i$

Possible values for further components/terms/addends

k_1	k_2	$k_1(c_1+d_1i)+k_2(-d_1+c_1i)$
0	0	$0+0i$
0	1	$-2-i$
1	0	$1-2i$
1	1	$-1+3i$

Consequently,

$$\begin{aligned} <\cdot|^{-}_{2+4i} = \{ & 0+0i, && -2+i, && 1+2i, && -1+3i; \\ & i, && -2+2i, && 1+3i, && -1+4i; \\ & 2i, && -2+3i, && 1+4i, && -1+5i; \\ & -1+i, && -3+2i, && 3i, && -2+4i; \\ & -1+2i, && -3+3i, && 4i, && -2+5i\}. \end{aligned}$$

Example 4. mod 3+6i, (case of 3).

$(3, 6)=3, \; c_1=1, \; d_1=2, \; \|p_1\|=5.$

Let us calculate p.s.a.n.v. from mod (3+6i). According to (1.13), we have:

$$x_1 + ix_2 = \frac{1}{5}(|2r|^{-}_{5} - 2r) + \frac{i}{5}(r+2\cdot|2r|^{-}_{5}) +$$
$$+ k_1(1+2i) + k_2(-2+i),$$
$$r = -2, -1, 0, 1, 2; \quad k_1, k_2 = -1, 0, 1.$$

Table of values $x_1^{(0)}, x_2^{(0)}$

r	$\lvert 2r\rvert^{-}_{5}$	$x_1^{(0)}$	$x_2^{(0)}$	$<\cdot\lvert^{-}_{1-2i}$
-2	1	1	0	1
-1	-2	0	-1	$-i$
0	0	0	0	0
1	2	0	1	i
2	-1	-1	0	-1

Possible values for further components/terms/addends

k_1	k_2	$k_1(c_1+xd_1i)+k_2(-d_1+c_1i)$
-1	-1	$1-3i$
-1	0	$-1-2i$
-1	1	$-3-i$
0	-1	$2-i$
0	0	$0+0i$
0	1	$-2+i$
1	-1	$3+i$
1	0	$1+2i$
1	1	$-1+3i$

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Consequently,

$$<\cdot\lvert^{-}_{3+6i} = \{ \begin{array}{rrrrr} 2-3i, & 1-4i, & 1-3i, & 1-2i, & -3i; \\ -2i, & -1-3i, & -1-2i, & -1-i, & -2-2i; \\ -2-i, & -3-2i, & -3-i, & -3, & -4; \\ 3-i, & 2-2i, & 2-i, & 2, & 1-i; \\ 1, & -i, & 0, & i, & -1; \\ -1+i, & -2, & -2+i, & -2+2i, & -3+i \\ 4+i, & 3, & 3+i, & 3+2i, & 2+i; \\ 2+2i, & 2+i, & 1+2i, & 1+3i, & 2i; \\ 3i & -1+2i, & -1+3i, & -1+4i, & -2+3i \end{array} \}.$$

P.s.n.v. and p.s.a.n.v. have simple geometric interpretation.

In view of formulas (1.7) deductions x+iy of p.s.n.v. mod (c+di) represent the integer points of the composite plane whose coordinates satisfy the system of the inequalities

$$0 \leqslant cx + dy < \| p \|,$$
$$0 \leqslant cy - dx < \| p \|.$$

In other words, p.s.n.v. on mod(c+di) with (c, d)=1 it is represented as the integer points, arranged/located within the square whose sides are assigned by the equations

$$cx + dy = 0, \quad cx - dy = \| p \|,$$
$$cy - dx = 0, \quad cy - dx = \| p \|,$$

including the apex/vertex of square with coordinates (0, 0).

Page 43. From theorem 2.1 it follows that with (c, d) =ρ ($\rho \neq 1$) to them it is supplemented on ρ-1 the integer points, arranged/located on two adjacent sides of the square

$$cx + dy = 0,$$
$$cy - dx = 0.$$

Let us note that the apexes/vertexes of square of p.s.n.v. on mod(c+di) have coordinates

$$A_1(0, 0), \; A_2(c, d), \; A_3(c - d, \; c + d), \; A_4(-d, \; c).$$

By force (1.7) for the elements/cells of p.s.n.v. according to mod(c+di) is valid the following evaluation:

$$\alpha + \beta i \in \langle \cdot \mid {}^{+}_{c+di},$$

$$\| \alpha + \beta i \| \leqslant \frac{2(\| p \| - 1)^2}{\| p \|}.$$

In figures 4-7 are given the images of p.s.n.v. on the moduli/modules of the given illustrative examples.

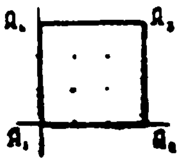

Fig. 4.

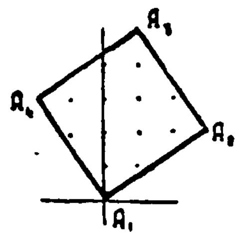

Fig. 5.

Fig. 4. P.s.n.v. on mod 3.

Fig. 5. P.s.n.v. on mod (3+2i).

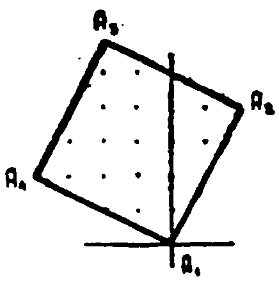

Fig. 6.

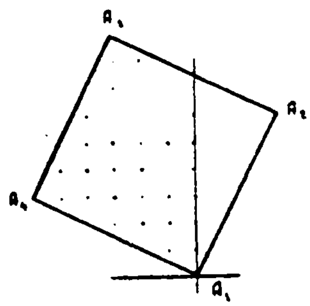

Fig. 7.

Fig. 6. P.s.n.v. on mod (2+4i).

Fig. 7. P.s.n.v. po mod (3+6i).

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Analogously deductions x+iy p.s.a.n.v. on mod(c+di) represent the integer points of the composite plane whose coordinates satisfy the system of the inequalities

$$-\frac{\|p\|}{2} \leqslant cx+dy < \frac{\|p\|}{2},$$

$$-\frac{\|p\|}{2} \leqslant cy-dx < \frac{\|p\|}{2}.$$

In other words, p.s.a.n.v. on mod(c+di) is represented by the integer points, arranged/located within the square whose sides are assigned by the equations

$$cx+dy=-\frac{\|p\|}{2},\ cx+dy=\frac{\|p\|}{2},$$

$$cy-dx=-\frac{\|p\|}{2},\ cy-dx=\frac{\|p\|}{2},$$

and it is also possibly, by the integer points, arranged/located on two adjacent sides

$$cx+dy=-\frac{\|p\|}{2},$$

$$cy-dx=-\frac{\|p\|}{2}.$$

A question about the possible presence of integer points on these sides requires special examination.

Let us preliminarily note only that the apexes/vertexes of square of p.s.a.n.v. on mod(c+di) have coordinates

$$A_1\left(\frac{c-d}{2}, \frac{c+d}{2}\right), A_2\left(-\frac{d+c}{2}, \frac{c-d}{2}\right),$$

$$A_3\left(\frac{d-c}{2}, \frac{-d+c}{2}\right), A_4\left(\frac{c+d}{2}, \frac{d-c}{2}\right),$$

and let us depict p.s.a.n.v. on the moduli/modules of the given above examples (Fig. 8-10).

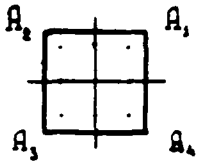

Fig. 8.

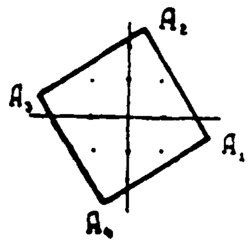

Fig. 9.

Fig. 8. P.s.a.n.v. on mod 3.

Fig. 9. P.s.a.n.v. on mod (3+2i).

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Let us consider, in what cases on the sides of square of p.s.a.n.v. are integer points. For ts.k.ch. just as for ts.v.ch., can be introduced the concept of parity and oddness of numbers.

Let be is preset ts.k.ch. c+di. It is obvious that $\|p\| \equiv 1 \pmod 2$ when and only when $c \not\equiv d \pmod 2$, and that $\|p\| \equiv 0 \pmod 2$ when and only when $c \equiv d \pmod 2$. At the same time in the second case it is possible to distinguish two subcases:

$$c \equiv d \equiv 0 \pmod 2 \text{ и } c \equiv d \equiv 1 \pmod 2.$$

Ts.k.ch. p=c+di odd, if $\|p\| \equiv 1 \pmod 2$, half-even, if $\|p\| \equiv 0 \pmod 2$

and $c\equiv d\equiv 1 \pmod 2$, even, if $\|p\|\equiv 0 \pmod 2$ and $c\equiv d\equiv 0 \pmod 2$.

It is possible to give the following two equivalent determinations.

1. Ts.k.ch. $p=c+di$ odd, if $\|p\|\equiv 1, 3 \pmod 4$, half-even, if $\|p\|\equiv 2 \pmod 4$, even, if $\|p\|\equiv 0 \pmod 4$.

K.ch. $p=c+di$ odd, if $p\equiv 1 \pmod{1-i}$, half-even, if $p\equiv 0 \pmod{1-i}$ and $c\equiv d\equiv 1 \pmod 2$, even, if $p\equiv 0 \pmod{1-i}$ and $c\equiv d\equiv 0 \pmod 2$.

Joining these properties, it is possible to conclude that if the norm of ts.k.ch. c+di is represented in the form $\|p\|=2^{\lambda}r$, where $r\equiv 1 \pmod 2$, then number c+di odd with $\lambda=0$, half-even with $\lambda=1$, even with $\lambda\geqslant 2$.

Hence, in particular, it follows that the product of two odd numbers is odd, the product of odd number by half-even is half-even, product of half-even numbers is even, the product even number to any ts.k.ch. is even.

Let us note that relative to addition are valid the following rules:

четное + четное = четное;
получетное + четное = получетное;
нечетное + четное = нечетное;
нечетное + получетное = нечетное;
получетное + получетное = четное;

$$\text{нечетное} + \text{нечетное} = \begin{cases} \text{получетное} \\ \text{четное.} \end{cases}$$

Key: (1). even. (2). half-even. (3). odd.

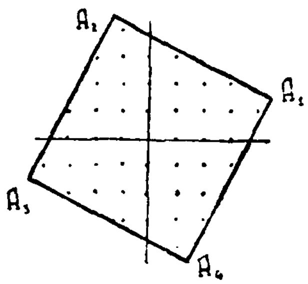

Fig. 10. P.s.n.v. on mod (3+6i).

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Theorem 2.2. If ts.k.ch. p=c+di - odd, then the sides of the square of p.s.a.n.v. contain not one integer point.

Proof. It suffices to examine one of the sides of the square of p.s.a.n.v. It is not difficult to see that the straight line, passing through the apexes/vertexes of square.

$$\left(\frac{c-d}{2}, \frac{c+d}{2}\right), \left(\frac{c+d}{2}, -\frac{c-d}{2}\right),$$

has the equation

$$cx+dy=\frac{c^2+d^2}{2}.$$

Since $c^2+d^2\equiv 1 \pmod 2$, then not one point with the whole coordinates can satisfy this equation.

Theorem 2.3. If p=c+di - half-even and (c, d)=1, then on the sides of square of p.s.a.n.v. there does not exist integer points,

with exception of the apexes/vertexes of square.

Proof. According to condition $c^2+d^2/2=k$ - ts.ch.

Let us consider equation of one of the sides of the square of p.s.a.n.v.

$$cx+dy=k.$$

Since $(c, d)=1$, the equation $cx+dy=1$ is solved in the integers. Let (x_0, y_0) - certain integral solution of this equation, then (kx_0, ky_0) be the solution of initial equation. In view of mutual simplicity of numbers c and d the general solution in the integers of homogeneous equation $cx+dy=0$ will take the form

$$x^*=dm,\ y^*=-cm,$$

where m - arbitrary ts.ch.

Therefore the general solution of initial equation can be represented in the form

$$x_m=kx_0+dm,$$

$$y_m=ky_0-cm.$$

Let us determine the distance between any two solutions

$$\sqrt{(x_m-x_r)^2+(y_m-y_r)^2}=|m-r|\sqrt{c^2+d^2}.$$

Hence it follows that the minimum distance between two different solutions of initial equation is equal to $\sqrt{c^2+d^2}$.

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Since integer points with the coordinates

$$\left(\frac{c-d}{2}, \frac{c+d}{2}\right), \left(\frac{c+d}{2}, -\frac{c-d}{2}\right)$$

(apex/vertex of square of p.s.a.n.v.) satisfy the equation in question and the distance between them is equal to $\sqrt{c^2+d^2}$, then between these two points on the straight line $cx+dy=c^2+d^2/2$ there does not exist integer points, which proves theorem.

Theorem 2.4. If ts.k.ch. $c+di$ is such, that $c+di\equiv0$ (mod $1-i$) and $(c, d)=\rho\,(\rho\neq1)$, then on each side of square of p.s.a.n.v. on mod$(c+di)$ is $p-1$, different from the apexes/vertexes, integer points.

Proof. Ts.k.ch. $c+di$, that satisfies condition $c+di\equiv0$ (mod $1-i$), relates to the class of half-even or even numbers and therefore it is represented in the form

$$c+di=2^{\lambda}\rho'(c_1+d_1 i),$$

where $(c_1, d_1)=1$, $\rho'\equiv1$ (mod 2). In this case $\lambda\geqslant1$, if c_1+d_1i - odd and $\lambda\geqslant0$, if c_1+d_1i - half-even. With $\lambda\geqslant1$ the assertion of theorem is corollary 2.1. With $\lambda=0$ for elements/cells $x+iy$ of p.s.a.n.v. on mod $\rho'(c_1+d_1i)$ in accordance with formulas (1.13) we have

$$x_1+ix_2=x_1^{(0)}+ix_2^{(0)}+k_1(c_1+d_1 i)+k_2(-d_1+c_1 i),$$

where

$$x_1^{(0)} = \frac{1}{\| p_1 \|}(c_1 \mid c_1^{-1} d_1 r \mid \bar{}_{\| p_1 \|} - d_1 r)$$

$$(\| p_1 \| = c_1^2 + d_1^2),$$

$$x_2^{(0)} = \frac{1}{\| p_1 \|}(c_1 r + d_1 \mid c_1^{-1} d_1 r \mid \bar{}_{\| p_1 \|})$$

i.e. deductions on the composite/compound modulus/module $\rho'(c_1+d_1i)$ are obtained by addition to each deduction on mod (c_1+d_1i) of the composite deductions of p.s.a.n.v. on mod ρ', represented to scale of the integral coordinate system with the orthogonal base $\{c_1+d_1i, -d_1+c_1i\}$. Consequently, since p.s.a.n.v. on mod (c_1+d_1i) is connected in their composition one of the apexes/vertexes of their square (in view of half-evenness of number c_1+d_1i), then in square of p.s.a.n.v. on mod $\rho'(c_1+d_1i)$ is placed $(\rho')^2$ squares of p.s.a.n.v. on mod (c_1+d_1i).

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Therefore on the sides of square of p.s.a.n.v. on mod $\rho'(c_1+d_1i)$ besides apexes/vertexes it will be placed $\rho'-1$ integer points.

As an example on Figures 11 and 12 are depicted the squares of p.s.a.n.v. on mod (9+3i) and mod (18+6i).

Finishing analysis of p.s.a.n.v., let us note, in the first place, that the elements/cells $\alpha+\beta i$ p.s.a.n.v. on mod p satisfy

evaluation $\|\alpha + \beta i\| \leqslant \frac{\|p\|}{2}$ (corollary of formulas (1.7), and, in the second place, p.s.a.n.v. in the best way satisfy the principle of tight packing of deductions in the sense that partial quotients m_1+m_2i, obtained as a result of dividing arbitrary ts.k.ch. a+bi into modulus/module c+di, satisfy in this case the evaluation

$$\left|\frac{a+bi}{c+di}-(m_1+im_2)\right|^2 \leqslant \frac{1}{2}.$$

Finally, let us point out to the possibility of use of ts.v.ch. as the elements/cells of p.s.v. on the composite modulus/module.

Theorem 2.5 (theorem of Gauss). If p=c+di(c, d≠0), moreover (c, d)=1, then $<\cdot|_p = |\cdot|_{\|p\|}$, in other words, p.s.v. on the composite modulus/module p=c+di can be defined as p.s.v. according to real modulus/module $\|p\|$, in this case has place isomorphism of the modular operations above the deductions on mod p and $\mathrm{mod}\,\|p\|$. if

$$<z\,|_p = |\,\underline{z}\,|_{\|p\|},$$

then

$$<w\,|_p = |\,\underline{w}\,|_{\|p\|},$$
$$<z+w\,|_p = |\,\underline{z}+\underline{w}\,|_{\|p\|},$$
$$<z\cdot w\,|_p = |\,\underline{z}\cdot\underline{w}\,|_{\|p\|}.$$

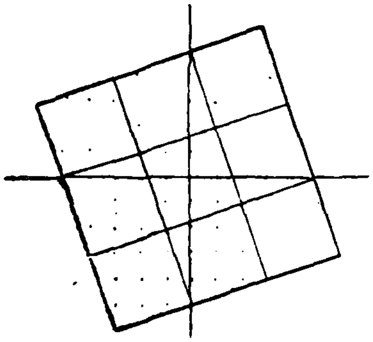

Fig. 11. Image of integer points on the sides of square of p.s.a.n.v. on mod (9+3i).

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Proof. Since (c, d)=1, there exist such ts.v.ch. λ, μ that $\lambda c+\mu d=1$.

Therefore $i=i(\lambda c+\mu d)=(\lambda d-\mu c)+(c+id)(\mu+i\lambda)$. Consequently, any ts.k.ch. z=a+bi is unambiguously represented in the form $z=a+b(\lambda d-\mu c)+p(b\mu-ib\lambda)$. Let us designate ts.v.ch. $\lambda d-\mu c$ through ρ. Let it be further

$$a+b\rho=|a+b\rho|_{\|p\|}+k(a, b)\|p\|.$$

Since $\|p\|=p\cdot\bar{p}$, then

$$z=|a+b\rho|_{\|p\|}+(k(a, b)(c-id)-b(\mu-i\lambda))p.$$

Since any ts.k.ch. z in a single manner is represented by the disintegration of this form, then

$$\cdot|_p=|\cdot|_{\|p\|},$$

moreover

$$< a + ib |_p = | a + \rho b |_p \, .$$

Let us note that if b=0, then

$$< a |_p = | a |_p \, .$$

Let us switch over to the proof of isomorphism.

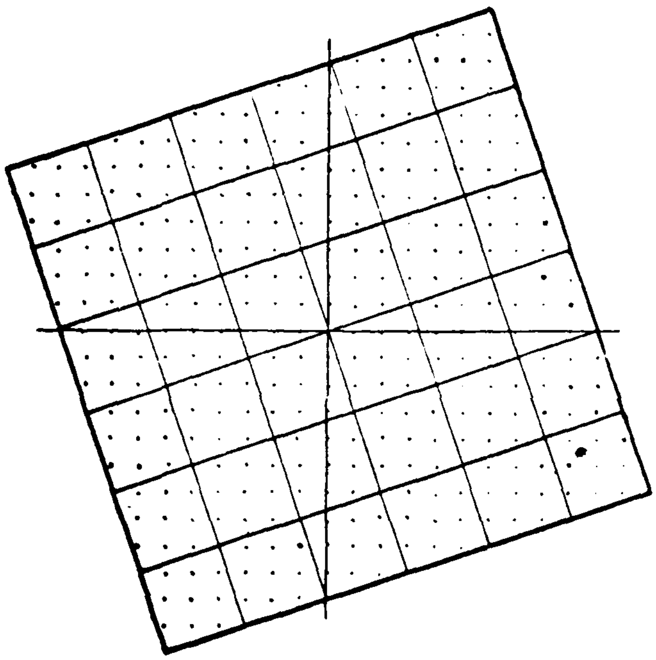

Fig. 12. Image of integer points on the sides of square of p.s.a.n.v. on mod (18+6i).

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We have

$$
\begin{gathered}
<(a_1+ib_1)+(a_2+ib_2)\,|_p=\ll a_1+ib_1\,|_p+<a_2+ib_2\,|_p\,|_p=\\
=<|\,a_1+\rho b_1\,|_{|p|}+|\,a_2+\rho b_2\,|_{|p|}\,|_p=\|\,a_1+\rho b_1\,|_{|p|}+\\
+|\,a_2+\rho b_2\,|_{|p|}\,|_{|p|}=|\,(a_1+\rho b_1)+(a_2+\rho b_2)\,|_{|p|}.
\end{gathered}
$$

It is analogous

$$
\begin{gathered}
<(a_1+ib_1)\,(a_2+ib_2)\,|_p=\ll a_1+ib_1\,|_p\cdot<a_2+ib_2\,|_p\,|_p=\\
=<|\,a_1+\rho b_1\,|_{|p|}\cdot|\,a_2+\rho b_2\,|_{|p|}\,|_p=\|\,a_1+\\
+\rho b_1\,|_{|p|}\cdot|\,a_2+\rho b_2\,|_{|p|}\,|_{|p|}=|\,(a_1+\\
+\rho b_1)\cdot(a_2+\rho b_2)\,|_{|p|}.
\end{gathered}
$$

Value ρ, obviously, is the function of modulus/module $p=c+di$; therefore we will write: $\rho=\rho(p)$.

Are valid the following properties. Property 1. $\rho(p)=\rho(i^s p)$, i.e. value ρ is invariant to all moduli/modules, associated with the data by modulus/module.

Property 2. $\rho(\bar{p})=-\rho(p)$, i.e., value of ρ on the conjugated/combined modulus/module is equal to the value of this modulus/module, undertaken with the opposite sign.

Proof. $\rho(p)=\lambda_0 d-\mu_0 c$, where (λ_0, μ_0) - the solution of the Diafantov equation $\lambda c+\mu d=1$. Let us consider modulus/module $ip=-d+ci$. Corresponding Diafantov equation accepts the form $-\lambda d+\mu c=1$, which has the solution

$$\lambda_1=-\mu_0,\ \mu=\lambda_0.$$

Consequently,

$$\rho(ip)=\lambda_1 c+\mu_1 d=-\mu_0 c+\lambda_0 d=\rho(p).$$

The Diafantov equation for the conjugated/combined modulus/module $\bar{p}=c-di$ has a form $\lambda c-\mu d=1$ and a solution $\bar{\lambda}=\lambda_0$, $\bar{\mu}=-\mu_0$. Consequently,

$$\rho(\bar{p})=-\bar{\lambda}d-\bar{\mu}c=-\lambda_0 d+\mu_0 c=-\rho(p).$$

Property 3. $\rho(p)$ - is the solution of comparison of $c+dx\equiv 0\ (\mathrm{mod}\ \|p\|)$.

Proof. The validity of this property is established by simple testing. In fact, let $x=\rho(p)=\lambda d-\mu c$, then

$$c+d(\lambda d-\mu c)=c+\lambda d^2-cd\mu=c+\lambda d^2-c+\\+c^2\lambda=\lambda(c^2+d^2)\equiv 0\ (\mathrm{mod}\ \|p\|).$$

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Table of modular additicns cn mcd(4+i).

+	0	1	2	−1−i	−i	1−i	2−i	−1−2i	−2i	2i	1+2i	−2+i	−1+i	i	1+i	−2	−1
0	0	1	2	−1−i	−i	1−i	2−i	−1−2i	−2i	2i	1+2i	−2+i	−1+i	i	1+i	−2	−1
1	1	2	−1−i	−i	1−i	2−i	−1−2i	−2i	2i	1+2i	−2+i	−1+i	i	1+i	−2	−1	0
2	2	−1−i	−i	1−i	2−i	−1−2i	−2i	2i	1+2i	−2+i	−1+i	i	1+i	−2	−1	0	1
−1−i	−1−i	−i	1−i	2−i	−1−2i	−2i	2i	1+2i	−2+i	−1+i	i	1+i	−2	−1	0	1	2
−i	−i	1−i	2−i	−1−2i	−2i	2i	1+2i	−2+i	−1+i	i	1+i	−2	−1	0	1	2	−1−i
1−i	1−i	2−i	−1−2i	−2i	2i	1+2i	−2+i	−1+i	i	1+i	−2	−1	0	1	2	−1−i	−i
2−i	2−i	−1−2i	−2i	2i	1+2i	−2+i	−1+i	i	1+i	−2	−1	0	1	2	−1−i	−i	1−i
−1−2i	−1−2i	−2i	2i	+2i	−2+i	−1+i	i	1+i	−2	−1	0	1	2	−1−i	−i	1−i	2−i
−2i	−2i	2i	1+2i	−2+i	−1+i	i	1+i	−2	−1	0	1	2	−1−i	−i	1−i	2−i	−1−2i
2i	2i	1+2i	−2+i	−1+i	i	1+i	−2	−1	0	1	2	−1−i	−i	1−i	2−i	−1−2i	−2i
1+2i	1+2i	−2+i	−1+i	i	1+i	−2	−1	0	1	2	−1−i	−i	1−i	2−i	−1−2i	−2i	2i
−2+i	−2+i	−1+i	i	1+i	−2	−1	0	1	2	−1−i	−i	1−i	2−i	−1−2i	−2i	2i	1+2i
−1+i	−1+i	i	1+i	−2	−1	0	1	2	−1−i	−i	1−i	2−i	−1−2i	−2i	2i	1+2i	−2+i
i	i	1+i	−2	−1	0	1	2	−1−i	−i	1−i	2−i	−1−2i	−2i	2i	1+2i	−2+i	−1+i
1+i	1+i	−2	−1	0	1	2	−1−i	−i	1−i	2−i	−1−2i	−2i	2i	1+2i	−2+i	−1+i	i
−2	−2	−1	0	1	2	−1−i	−i	1−i	2−i	−1−2i	−2i	2i	1+2i	−2+i	−1+i	i	1+i
−1	−1	0	1	2	−1−i	−i	1−i	2−i	−1−2i	−2i	2i	1+2i	−2+i	−1+i	i	1+i	−2

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The practical value cf thecrem 2.5 lias in the fact that it with sufficiently wide limitations on the moduli/mcdules establishes/installs the structural adequacy of the tabl⁻s of modular operations on the composite and real moduli/mcdules, thanks to which appears the possibility, in the first place, circuit realization of

one and the same table to use both with the work with the real and composite moduli/modules, secondly the properties of symmetry, inherent in p.s.v. on the composite moduli/modules, to extend on p.s.v. on the real moduli/modules.

Let us give an illustrative example. Let $p=4+i$, $\|4+i\|=17$. Here $c=4$, $d=1$. The solution of the equation

$$4\lambda+\mu=1$$

can be selected thus:

$$\lambda=1,\ \mu=-3.$$

In that case

$$\rho=\lambda d-\mu c=1\cdot 1-(-3)\cdot 4=13.$$

As p.s.v. on mod(4+i) let us select p.s.a.n.v., while as p.s.v. on mod 17 - p.s.n.v. Then the correctly isomorphic relation

$$< x+iy \,|^{-}_{4+i} \longleftrightarrow |\, x-13y\,|_{17},$$

which is described by the following table;

$\lvert\cdot\rvert^{+}_{17}$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$<\cdot\rvert^{-}_{4+i}$	0	1	2	$-1-i$	$-i$	$1-i$	$2-i$	$-1-2i$	$-2i$	$2i$	$1+2i$	$-2+i$	$-1+i$	i	$1-i$	-2	-1

Isomorphism is clearly illustrated by the tables of modular additions.

Table of modular additions on mod 17.

+	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
0	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
1	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	0
2	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	0	1
3	3	4	5	6	7	8	9	10	11	12	13	14	15	16	0	1	2
4	4	5	6	7	8	9	10	11	12	13	14	15	16	0	1	2	3
5	5	6	7	8	9	10	11	12	13	14	15	16	0	1	2	3	4
6	6	7	8	9	10	11	12	13	14	15	16	0	1	2	3	4	5
7	7	8	9	10	11	12	13	14	15	16	0	1	2	3	4	5	6
8	8	9	10	11	12	13	14	15	16	0	1	2	3	4	5	6	7
9	9	10	11	12	13	14	15	16	0	1	2	3	4	5	6	7	8
10	10	11	12	13	14	15	16	0	1	2	3	4	5	6	7	8	9
11	11	12	13	14	15	16	0	1	2	3	4	5	6	7	8	9	10
12	12	13	14	15	16	0	1	2	3	4	5	6	7	8	9	10	11
13	13	14	15	16	0	1	2	3	4	5	6	7	8	9	10	11	12
14	14	15	16	0	1	2	3	4	5	6	7	8	9	10	11	12	13
15	15	16	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
16	16	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15

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In conclusion let us note that in contrast to real numbers for complex numbers there are p.s.v. consisting of an equal quantity of nonisomorphic between themselves deductions.

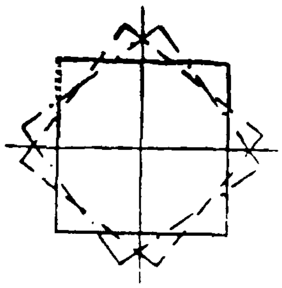

Fig. 13. Squares of p.s.a.n.v.: (-) - on mod 5; (-•-) - on mod 3+4i; (- -) - on mod 3-4i.

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Example. Let us consider p.s.v. on the moduli/modules 3+4i, 3-4i, 5. A quantity of elements/cells of p.s.v. on each of these moduli/modules is equal to 25 (Fig. 13).

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PRINCIPLES OF THE MACHINE ARITHMETIC OF COMPLEX NUMBERS.(U)
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UNCLASSIFIED FTD-ID(RS)T-0240-81 NL

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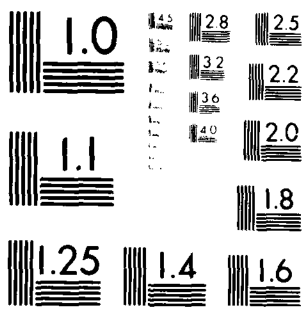
1.0
1.1
1.25
1.4
1.6
1.8
2.0
2.2
2.5
2.8
3.2
3.6
4.0
4.5

$<\cdot\|_{3+4i}$	$<\cdot\|_{3-4i}$	$<\cdot\|_{5}$
$-3i$	$-3i$	$-2-2i$
$-1-2i$	$-1-2i$	$-1-2i$
$-2i$	$-2i$	$-2i$
$1+2i$	$1+2i$	$1-2i$
$-2-i$	$-2-i$	$2-2i$
$-1-i$	$-1-i$	$-2-i$
$-i$	$-i$	$-1-i$
$1-i$	$1-i$	$-i$
$2-i$	$2-i$	$1-i$
-3	-3	$2-i$
-2	-2	-2
-1	-1	-1
0	0	0
1	1	1
2	2	2
3	3	$-2+i$
$-2+i$	$-2+i$	$-1+i$
$-1+i$	$-1+i$	$+i$
i	i	$1+i$
$1+i$	$1+i$	$2+i$
$2+i$	$2+i$	$-2+2i$
$-1+2i$	$-1+2i$	$-1+2i$
$2i$	$2i$	$2i$
$1+2i$	$1+2i$	$1+2i$
$3i$	$3i$	$2+2i$

$-2+2i \in <\cdot|_{5},\ 2-i \in <\cdot|_{5}$

and therefore

$$-2+2i \not\equiv 2-i \pmod 5.$$

But

$$-2+2i-(2-i) = -4+3i = i(3+4i),$$

therefore

$$-2+2i \equiv 2-i \pmod{3+4i}$$

and, conversely,

$$3i \in <\cdot|_{3+4i},\ -2i \in <\cdot|_{3+4i},$$

i.e.

$$3i \not\equiv -2i \pmod{3+4i},$$

but

$$3i \equiv -2i \pmod 5.$$

Thus, there are deductions of p.s.v. on mod(3+4i), congruent between themselves in mod5, and in mod5, congruent between themselves on mod(3+4i). On the other hand, p.s.a.n.v. in mod(3+4i) and mod(3-4i) simply coincide, and, at the same time, between these p.s.v. there does not exist isomorphism.

Actually/really,

$$<(-2-i)+(1-i)\,|_{3+4i}=2i,$$

on the other hand,

$$<(-2-i)+(-1-i)\,|_{3-4i}=1+i.$$

§3. Other methods of the assignment of the full/total/complete systems of deductions.

The development of the machine algorithms of complex numbers requires the refinements of the concept about the range.

The method of coding ts.k.ch. and the algorithm of the arithmetic operations on ts.k.ch. depicted as the preset code, is called numeration system.

Range - this the final (but therefore discrete/digital) set of complex numbers, with which uses the computer.

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In view of discreteness any range can be depicted as the set of ts.k.ch. The method of the assignment of range depends substantially on the method of the coding of ts.k.ch. . . . It is more exact, the form of coding of ts.k.ch. accepted determines that set of the numbers, with which in the final analysis uses the computer.

Of the method of coding is required

a) one-to-one image of integral quantities by the preset code;

b) the realizability of all arithmetic operations in the language of this code;

c) the presence of the sign/criterion by which it would be possible to judge about the output/yield for the range of the result of arithmetic operation.

From the point of view of simplicity the greatest interest is represented by the methods of the assignment of range of p.s.v. with respect to the composite/compound modulus/module.

In connection with this let us note three most important methods of assignment of p.s.v. with respect to composite/compound modulus/module.

Theorem 3.1. The set of ts.k.ch. of the form

$$\xi_0+\xi_1 p+\xi_2 p^2+\ldots+\xi_{n-1} p^{n-1}, \qquad (3.1)$$

where $\xi_k (0\leqslant k\leqslant n-1)$ pass values of p.s.v. $\langle\cdot|_p$, it forms p.s.v. $\langle\cdot|_{p^n}$.

Proof. In view of theorem 2.3.1 it suffices to show that any ts.k.ch. z in a single manner is represented in the form

$$z=\zeta+q\cdot p^n,$$

where ζ - ts.k.ch. form (3.1).

For this purpose we will use the following algorithm.

Let us present number z in the form $z=\xi_0+q_1 p$, where $\xi_0\in\langle\cdot|_p$. Here ξ and q_1 are determined unambiguously, since $\langle\cdot|_p$ - p.s.v. on mod p.

Q_1 we relatively act analogously. We have $q_1=\xi_1+q_2 p$, where ξ_1, q_2 is newly determined unambiguously, since $\xi_1\in\langle\cdot|_p$.

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As a result we will have a chain/network of the equalities

$$\begin{aligned} z &= \xi_0 + q_1 p, & \xi_0 &\in \langle\cdot\,|_p, \\ q_1 &= \xi_1 + q_2 p, & \xi_1 &\in \langle\cdot\,|_p, \\ &\cdots\cdots & &\cdots\cdots \\ q_k &= \xi_k + q_k p, & \xi_k &\in \langle\cdot\,|_p, \\ &\cdots\cdots & &\cdots\cdots \\ q_{n-1} &= \xi_{n-1} + q_n p, & \xi_{n-1} &\in \langle\cdot\,|_p, \end{aligned}$$

whence arbitrary ts.k.ch. in a single manner is represented in the form

$$z = \xi_0 + \xi_1 p + \xi_2 p^2 + \ldots + \xi_{n-1} p^{n-1} + q^n p^n.$$

Theorem 3.2. The set of ts.k.ch. of the form

$$\xi_1 + \xi_2 p_1 + \xi_3 p_1 p_2 + \xi_4 p_1 p_2 p_3 + \ldots + \xi_n p_1 p_2 \ldots p_{n-1},$$

where $\xi_k (1 \leqslant k \leqslant n)$ respectively passes p.s.v. $\langle\cdot\,|_p$, it forms p.s.v. on modulus/module $P(P = p_1 p_2 \ldots p_n)$.

Proof is analogous. In this case we will have a chain/network of the equalities

$$\begin{aligned} z &= \xi_1 + q_1 p_1, & \xi_1 &\in \langle\cdot\,|_{p_1}, \\ q_1 &= \xi_2 + q_2 p_2, & \xi_2 &\in \langle\cdot\,|_{p_2}, \\ &\cdots\cdots & &\cdots\cdots \\ q_{n-1} &= \xi_n + q_n p_n, & \xi_n &\in \langle\cdot\,|_{p_n}, \end{aligned}$$

whence

$$z = \xi_1 + \xi_2 p_1 + \ldots + \xi_n p_1 p_2 \ldots p_{n-1} + q_n p_1 p_2 \ldots p_n.$$

Theorem 3.3. Let $p_1, p_2, \ldots, p_n$ - ts.k.ch. pair-wise mutually simple, then the set of ts.k.ch. of the form

$$\sum_{k=1}^{n} \alpha_k m_k P_k, \qquad (3.3)$$

where $\alpha_k (1 \leqslant k \leqslant n)$ pass values of p.s.v. $<\cdot|_{p_k}$ respectively; $P_k = p_1 p_2 \ldots p_{k-1} p_{k+1} \ldots p_n$ and $<P_k^{-1}|_{p_k} = m_k$ - deductions on modulus/module p_k such, which

$$\ll m_k \cdot < P_k|_{p_k}|_{p_k} = 1$$

forms of p.s.v. on composite/compound modulus/module $P = p_1 p_2 \ldots p_n$.

Proof will consist of two stages: first, which any ts.k.ch. is congruent in mod by P with one of the numbers of form (3.3), in the second place, that numbers of form (3.3) are incomparable between themselves on mod P.

Let z - arbitrary ts.k.ch. and

$$z \equiv \alpha_k (\mathrm{mod}\, p_k),\ \alpha_k \in <\cdot|_{p_k},\ 1 \leqslant k \leqslant n. \qquad (3.4)$$

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We form the number

$$\zeta = \sum_{k=1}^{n} \alpha_k < P_k m_k|_{p_k} \cdot P_k.$$

Since ts.k.ch. $m_k P_k$ are such, that

$$< m_k P_k|_{p_j} = \begin{cases} 0, \text{ если}^{(1)}\ j \neq k, \\ 1, \text{ если}\ j = k, \end{cases} \qquad (3.5)$$

Key: (1). if.

then

$$\zeta \equiv \alpha_k \pmod{p_k} \quad 1 \leqslant k \leqslant n.$$

By force (3.4)

$$z \equiv \zeta \pmod{p_k} \quad 1 \leqslant k \leqslant n.$$

But, according to property (2.1.4), it follows that

$$z \equiv \zeta \pmod{P}.$$

Let us further consider two arbitrary different combinations of the deductions

$$\begin{array}{ll} (\alpha_1, \alpha_2, \ldots, \alpha_n) & \alpha_k \in <\cdot|_{p_k} \\ & \qquad 1 \leqslant k \leqslant n, \\ (\beta_1, \beta_2, \ldots, \beta_n) & \beta_k \in <\cdot|_{p_k} \end{array}$$

moreover, at least, at one value k: $\alpha_k = \beta_k$. Corresponding to them ts.k.ch. of form (3.3) are such:

$$\zeta_1 = \sum_{k=1}^{n} \alpha_k < P_k^{-1} |_{p_k} P_k,$$

$$\zeta_2 = \sum_{k=1}^{n} \beta_k < P_k^{-1} |_{p_k} P_k.$$

Let us show that $\zeta_1 \not\equiv \zeta_2$ (mod P). Let us assume reverse that $\zeta_1 = \zeta_2$ (mod P), then according to property (2.1.5)

$$\zeta_1 \equiv \zeta_2 \pmod{p_k}, \quad 1 \leqslant k \leqslant n.$$

hence by force (3.5) we have

$$\alpha_k \equiv \beta_k \pmod{p_k}, \quad 1 \leqslant k \leqslant n,$$

however, since $\alpha_k, \beta_k \in <\cdot|_{p_k}$, that

$$\alpha_k = \beta_k \qquad (1 \leqslant k \leqslant n),$$

which contradicts assumption.

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The theorems of the theory of numbers indicated include three most important methods of the coding of ts.k.ch. The first theorem leads to the positional numeration system with the composite basis/base, the second - to the positional numeration system with the mixed radices (to the so-called, polyadic code), to the nonpositional numeration system in the residual classes. All these numeration systems will be traced below. Let us note only some special features/peculiarities in the geometric image of p.s.v. determined by these theorems. These special features/peculiarities are connected with the fact that the numeration systems of complex numbers carry planar character.

Let p=c+di - ts.k.ch. It is possible to give the following geometric interpretation of the product of two ts.k.ch.

Is correct the identity

$$(a+bi)(c+di)=a(c+di)+b(-d+ci).$$

Hence it follows that product of two ts.k.ch. (a+bi) and (c+di) is represented as integer point with the coordinates (a, b) to scale of the integral system of coordinates of orthogonal base {c+di, -d+ci}.

Let us introduce designation $p'=ip=-d+ci$. Then positional recording of ts.k.ch. z

$$z=\xi_0+\xi_1 p+\xi_2 p^2+\ldots+\xi_{n-1} p^{n-1},$$

where $\xi_k=x_k+iy_k \in \langle\cdot\,|_p$ can be represented in the form

$$z=x_0+iy_0+(x_1 p+y_1 p')+(x_2 p^2+y_2 (p^2)')+$$
$$+\ldots+(x_{n-1} p^{n-1}+y_{n-1} (p^{n-1})').$$

where each component/term/addend

$$\xi_k p^k=x_k p^k+y_k (p^k)'$$

as representative point with coordinates (x_k, y_k) to scale of integral system of coordinates of orthogonal base $\{p^k, (p^k)'\}$.

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Thus, the vector sense of digit $\xi_k=(x_k, y_k)$ of the k digit of positional recording of ts.k.ch. lies in the fact that numbers x_k, y_k are the coordinates of vector in base $\{p^k, (p^k)'\}$, and number itself z

$$z=(x_0, y_0; x_1, y_1; x_2, y_2; \ldots; x_{n-1}, y_{n-1})$$

is the sum of the vectors of the given ones by its coordinates $(x_0, y_0), (x_1, y_1), (x_2, y_2), \ldots, (x_{n-1}\ y_{n-1})$ respectively in the systems of coordinates

$$(1, i), (p, p')\ (p^2, (p^2)') \ldots, (p^{n-1}, (p^{n-1})').$$

Example 1. Let us consider modulus/module (basis/base of the

positional recording of a number) p=1+2i. As the digits of the positional code let us select p.s.a.n.v. on mod(1+2i)

$$<\cdot|\overline{\ }_{1+2i} = \{0, 1, i, -1, -i\}.$$

The image of the number

$$1 + (-i)(1 + 2i) + (-1)(1 + 2i)^2 = 1 + (0\cdot p + (-1)p') + ((-1)p^2 + 0\cdot(p^2)') = 6 - 5i$$

is shown in figure 14.

The diversity of the configurations of ranges, generated by the positional code of complex numbers, is illustrated by the following examples.

Example 2. Figure 15 and 16 depicts p.s.v. on mod $(1+2i)^2$ and mod $(1+2i)^3$, represented by the respectively positional code

$$\xi_0 + \xi_1(1 + 2i),$$
$$\xi_0 + \xi_1(1 + 2i) + \xi_2(1 + 2i)^2,$$

where $\xi_k \in <\cdot|\overline{\ }_{1+2i}$ $(k = 0, 1, 2)$.

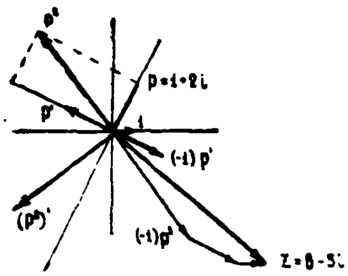

Fig. 14.

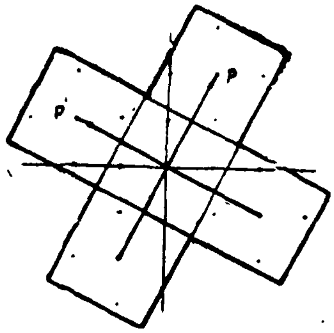

Fig. 15.

Fig. 14. Representation of number $z=1+(0\cdot p+(-1)p^1+(i-1)p^2+0\cdot(p^2)^1)=6-5i$.

Fig. 15. P.s.v. on mod$(1+2i)^2$.

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Example 3. On figures 17 and 18 are depicted p.s.v. on mod$(1+2i)^2$ and mod$(1+2i)^3$, represented by the positional of the codes

$$\zeta_0+\zeta_1(1+2i),\ \zeta_0+\zeta_1(1+2i)+\zeta_2(1+2i)^2,$$

where $\zeta_k \in \langle \cdot \mid \overset{+}{1-2i}$

Example 4. On figures 19 and 20 are depicted p.s.v. on mod$(1+2i)^2$ and mod$(1+2i)^3$ represented by the positional code

$$\zeta_0+\zeta_1(1-2i),\ \zeta_0+\zeta_1(1-2i)+\zeta_2(1-2i)^2,$$

where $\zeta_k \in \cdot \mid \cdot \mid_{5}^{-}$.

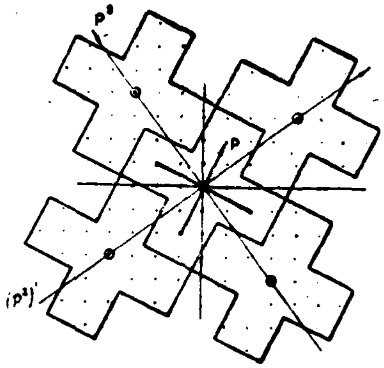

Fig. 16.

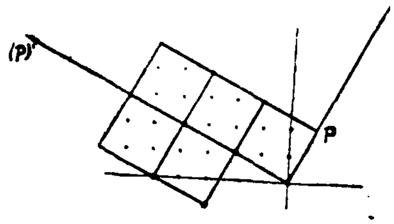

Fig. 17.

Fig. 16. P.s.v. on mod$(1+2i)^3$.

Fig. 17. P.s.v. on mod $(1+2i)^2$.

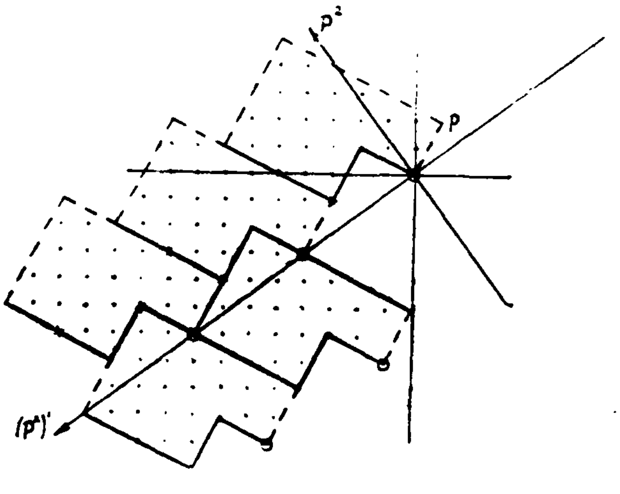

Fig. 18. P.s.v. on mod$(1+2i)^3$.

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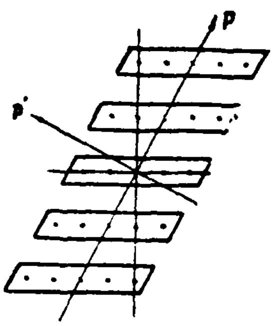

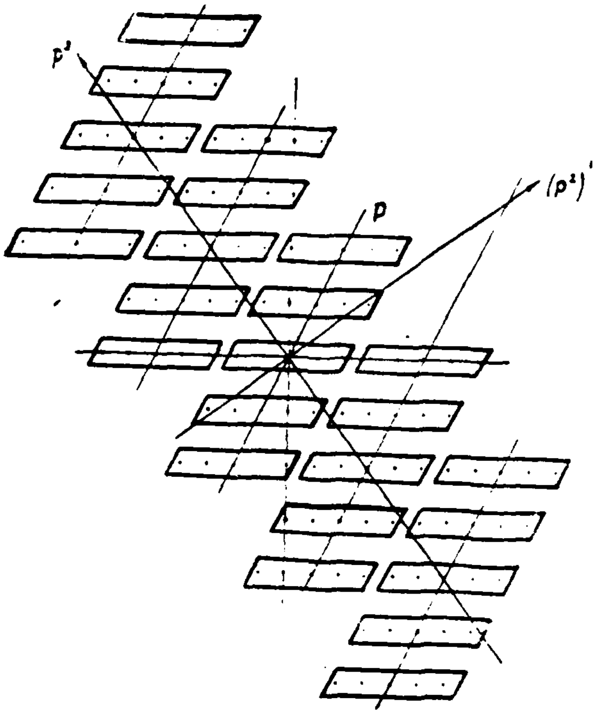

Fig. 19. Fig. 20.

Fig. 19. P.s.v. on mod$(1+2i)^2$.

Fig. 20. P.s.v. on mod$(1+2i)^3$.

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Relative to the mixed codes of p.s.v. on the composite/compound

modulus/module let us note that the described principle of image of p.s.v. completely is retained (Fig. 21).

The special feature/peculiarity of the mixed codes in contrast to the appropriate codes for real numbers is the fact that here, generally speaking it is broken the requirement of invariance. The essence of this requirement consists of the following.

If for real numbers is satisfied the condition: p.s.v. on

$$\mathrm{mod}\, P\ (P = p_1 p_2, \ldots, p_n),$$

determined by the mixed positional of the codes

$$x_1 + x_2 p_1 + x_3 p_1 p_2 + \ldots + x_n p_1 \ldots p_{n-1},$$

where x_n passes p.s.n.n.v. or p.s.a.n.v., it remains invariant relative to sequence of moduli/modules $p_1, p_2, \ldots, p_n$, the for complex numbers this requirement in general proves to be incomplete. This fact is illustrated by figures 21 and 22, where they are depicted p.s.v. on composite/compound modulus/module (1+2i) (1+3i). In this case figure 21 depicts p.s.v. determined by the code $\varepsilon_1 + \varepsilon_2(1+3i)$, where

$$\xi_1 \in <\cdot|_{\overline{1+3i}},$$
$$\xi_2 \in <\cdot|_{\overline{1+2i}},$$

and figure 22 depicts p.s.v. determined by the code $\varepsilon_1 + \varepsilon_2(1+2i)$, where

$$\xi_1 \in <\cdot|_{\overline{1+2i}},$$
$$\xi_2 \in <\cdot|_{\overline{1+3i}}.$$

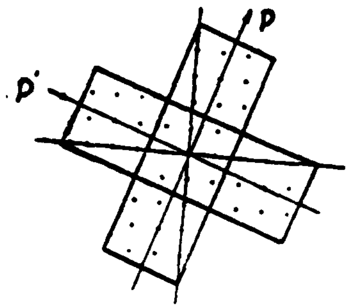

Fig. 21.

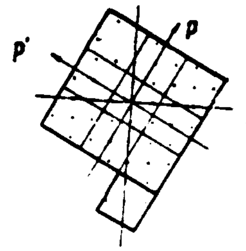

Fig. 22.

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The absence of invariance substantially complicates rounding in the nonpositional arithmetic of complex numbers.

In chapter 5 will be examined one version of the invariant mixed positional code for a complex number. In connection with the construction of such codes it is expedient to consider, in what relationship/ratio is located the full/total/complete system of absolutely smallest composite deductions through $\|p\|$ and p.s.v. determined by mixed positional code $\xi+\eta p$, where $\xi\in\langle\cdot|_{p}^{-}$, $\eta\in\langle\cdot|_{\bar p}^{-}$.

Theorem 3.4. Let $p=c+di\ (c,\ d\neq0)\ (\|p\|>2)$. Any ts.k.ch. $z\in\langle\cdot|_{\|p\|}^{-}$ is decomposed/expanded on bases/bases p, $\bar p$ into the mixed positional code of the form

$$z=\xi+\eta p+\varepsilon\|p\|.$$

where

$$\xi \in \langle\cdot|\overline{_p}, \eta \in \langle\cdot|\overline{_p}, \|\epsilon\| = \begin{cases} 0 \\ 1. \end{cases}$$

Proof. Let us compare p.s.v. determined by the code

$$\xi + \eta p \quad (\xi \in \langle\cdot|\overline{_p}, \eta \in \langle\cdot|\overline{_p}),$$

with p.s.a.n.v. $\langle\cdot|\overline{_{1,p}}$. For this purpose let us show that the square of p.s.a.n.v. in mod p, examined/considered to scale of the integral system of coordinates of base {p, p'}, coincides with the square of p.s.a.n.v. $\langle\cdot|\overline{_{1,p}}$. The apexes/vertexes of the square in question in the system {p, p'} have the following coordinates:

$$\left(\frac{c+d}{2}, \frac{c-d}{2}\right), \left(-\frac{c-d}{2}, \frac{c+d}{2}\right); \left(-\frac{c+d}{2}, -\frac{c-d}{2}\right);$$
$$\left(\frac{c-d}{2}, -\frac{c+d}{2}\right).$$

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In the recalculation upon the integral grid of base {1, i} these points take the form

$$\frac{c+d}{2}p + \frac{c-d}{2}p' = \frac{\|p\|}{2}(1+i),$$
$$-\frac{c-d}{2}p + \frac{c+d}{2}p' = \frac{\|p\|}{2}(-1+i),$$
$$-\frac{c+d}{2}p - \frac{c-d}{2}p' = \frac{\|p\|}{2}(-1-i),$$
$$\frac{c-d}{2}p - \frac{c-d}{2}p' = \frac{\|p\|}{2}(1-i).$$

But latter/last points are the apexes/vertexes of square of

p.s.a.n.v. $<\cdot|^{-}_{\|p\|}$. Further by the cut square $<\cdot|^{-}_{\|p\|}$ by the grid of the squares, congruent to square $<\cdot|^{-}_{p}$, let us show that the apexes/vertexes of square $<\cdot|^{-}_{\|p\|}$, belong to this network. Actually/really, the family of the orthogonal straight lines of network is determined by the equations

$$cx+dy=\frac{\lambda}{2}\|p\|,$$

$$-dx+cy=\frac{\mu}{2}\|p\|.$$

Latter/last system with

$$\lambda=c+d,\ \mu=c-d,\ \lambda=d-c,\ \mu=c+d;$$

$$\lambda=-c-d,\ \mu=d-c,\ \lambda=c-d,\ \mu=-c-d,$$

has respectively solutions

$$\left(\frac{\|p\|}{2},\ \frac{\|p\|}{2}\right),\ \left(-\frac{\|p\|}{2},\ \frac{\|p\|}{2}\right),\ \left(-\frac{\|p\|}{2},\ -\frac{\|p\|}{2}\right),$$

$$\left(\frac{\|p\|}{2},\ -\frac{\|p\|}{2}\right).$$

However, set of ts.k.ch. covered/coated with the squares of the network in question whose centers are elements/cells of p.s.a.n.v. on mod $\bar{p}$, preset in scale of the integral system of coordinates of base {p, p'}, is formed p.s.v. described by the code

$$\xi+\eta p\ (\xi\in<\cdot|^{-}_{p},\ \eta\in<\cdot|^{-}_{\bar{p}}).$$

Since in this case the apexes/vertexes of square $<\cdot|^{-}_{\|p\|}$ belong to network, then output/yield for square $<\cdot|^{-}_{\|p\|}$ of numbers of form $\xi+\eta p$ is feasible only in the direction of the axes of the coordinates of base {p, p'}. The distance of these points from the sides of square

$<\cdot|^{-}_{p}$ cannot exceed value $\frac{\sqrt{2}}{2}|p|$.

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Since inequality $\frac{\sqrt{2}}{2}|p|+\|p\|<2\|p\|$ is fulfilled for all ts.k.ch. that satisfy condition $\|p\|>2$, then the difference between numbers $\xi+\eta p$, left beyond the limits of square $<\cdot|^{-}_{p}$, and their composite least positive residues z in mod $\|p\|$ it is determined by equality $z-(\xi+\eta p)=\varepsilon\|p\|$, where $\|\varepsilon\|=1$.

For all numbers of means $\xi+\eta p$, of those not emerging beyond limits $<\cdot|^{-}_{p}$, is obvious $\varepsilon=0$. Theorem is illustrated by examples.

Example. p=1+2i.

$$<\cdot|^{-}_{1+2i}=\{0,\ 1,\ 2,\ -1,\ -2,\ i,\ 2i,\ -i,\ -2i,\ 1+i,\ 2+i,\ 1+2i,\ 2+2i,\ -1+i,\ -2+i,\ -1+2i,\ -2+2i,\ -1-i,\ -1-2i,\ -2-i,\ -2-2i,\ 1-i,\ 2-i,\ 1-2i,\ 2-2i\};$$

a) ts.k.ch. - $-2+2i\in<\cdot|^{-}_{1+2i}$ and its disintegration into the mixed positional code take the form

$$-2+2i=i-i(1-2i)+0\cdot 5,\ \text{т. е.}\ \ldots=0;$$

Key: (1). i.e.

b) ts.k.ch. $1-2i\in<\cdot|^{-}_{1+2i}$, its disintegration is such:

$$1-2i=i+1\cdot(1+2i)+(-i)(5)\ \text{и}\ \ldots-i\ =1;$$

Key: (1). and.

c) ts.k.ch. $-2-i \in \langle \cdot | \bar{\cdot}_{1-2i} \rangle$. its disintegration takes the form $-2-i = 1 + (-i)(1+2i) + (-1)$ (5); therefore $[\cdot] = 1$.

§4. Some special features/peculiarities of the full/total/complete system of deductions.

Modular numbers. During the generalization of numeration system to the composite bases/bases appears one special feature/peculiarity, which depending on selection of p.s.v. can have an essential effect on the organization of the processes of transfer with the execution of the arithmetic operations above numbers, given by the positional code.

Let us consider the elucidating example. Let it be preset basis/base p=2+i and as p.s.v. chosen p.s.n.v.

$$\langle \cdot |_p^+ = \{0, i, 1-i, 2i, 1-2i\}.$$

It is not difficult to check the validity of the representations

$$-1 = (-1)(2-i) - (1-i), \quad (1-i \in \langle \cdot |_p^+),$$

$$-1-i = (-1-i)(2+i) + 2i, \quad (2i \in \langle \cdot |_p^+).$$

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Hence it follows that numbers -1 and (-1-i) can be depicted as the

positional code of how conveniently large length n, since with arbitrary n, since with arbitrary n are valid the equalities

$$-1=(1+i)+(1+i)p+(1+i)p^2+\ldots+(1+i)p^n+(-1)p^{n+1},$$
$$-1-i=2i+2ip+2ip^2+\ldots+2ip^n+(-1-i)p^{n+1}.$$

Further, since a product of two deductions i•i=-1, and number -1 is represented as the positional code of the arbitrary length n with basis/base p=2+i, then this complicates the process of transfer with the execution of multiplication. As can easily be seen from the given tables, the process of addition in the case in question does not suffer the deficiency/lack indicated.

Table of summation on mod(2+i).

+	0	i	$1+i$	$2i$	$1+2i$
0	0	i	$1+i$	$2i$	$1+2i$
i	i	$2i$	$1+2i$	$1+i$	0
$1-i$	$1+i$	$1+2i$	i	0	$2i$
$2i$	$2i$	$1+i$	0	$1+2i$	i
$1+2i$	$1+2i$	0	$2i$	i	$1+i$

Table of transfers with the summation over mod(2+i).

π	0	i	$1+i$	$2i$	$1+2i$
0	0	0	0	0	0
i	0	0	0	i	$1+i$
$1+i$	0	0	1	$1+i$	1
$2i$	0	i	$1+i$	i	$1-i$
$1+2i$	0	$1+i$	1	$1+i$	$1+i$

Observation. Since $-1=1+(-1)2$, then analogous situation appears in the binary arithmetic of real numbers, if we the sign of a number

do not select by the independent sign multiplier.

In connection with those presented it is of interest to consider the question: in what cases there are numbers x+iy, which satisfy the condition

$$x+iy=(x+iy)p+\alpha+i\beta, \text{ where } \alpha+i\beta \in <\cdot|_p. \quad (4.1)$$

This number, if they exist, we will call by modular on mod p. Equality (4.1) is equivalent to system of equations

$$x = xc - dy + \alpha,$$

$$y = cy + dx + \beta.$$

Solving this system, we find

$$x = \frac{1}{\Delta}(-\alpha(c-1) - \beta d),$$

$$y = \frac{1}{\Delta}(-\beta(c-1) + \alpha d),$$

where

$$\Delta = (c-1)^2 + d^2. \quad (4.2)$$

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On the other hand, we note that

$$\frac{-\alpha-\beta i}{(c-1)+di} = \frac{-\alpha(c-1)-\beta d}{\Delta} + i\frac{-\beta(c-1)+\alpha d}{\Delta}.$$

Hence we consist.

Theorem 4.1. Modular numbers on mod p exist when and only when p.s.v. (mod p) are included deductions $\alpha+\beta i$ such, that

$$\alpha+\beta i \equiv 0 \pmod{p-1}.$$

Thus, the presence of modular numbers depends substantially on the selection of p.s.v. on mod p. Trace in this plan/layout the full/total/complete systems of the smallest and least positive residues.

For simplicity of reasonings we will be converted to the geometric illustrations (Fig. 23-26).

Case 1. For p.s.n.v. on mod(c+di), where c>0, there is at least one modular number. This follows from the fact that with c>0 always

$$(c-1)+di \in <\cdot|^{+}_{c+di}.$$

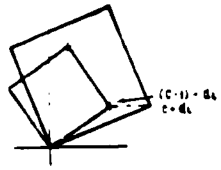

Fig. 23.

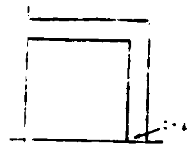

Fig. 24.

Fig. 23. c≥2, d>0.

Fig. 24. c≥2, d=0.

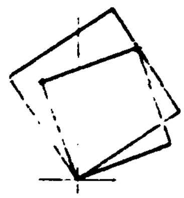

Fig. 25.

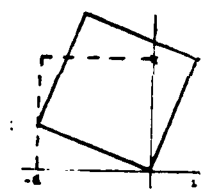

Fig. 26.

Fig. 25. c+di.

Fig. 26. c=1, d>0.

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Let us note that for p.s.n.v. $<\cdot|_2^+=\{0,1\ i,1+i\}$ are two modular numbers: by -1 and -i, since

$$-1=(-1)2+1, \quad (1 \in <\cdot|_{2}^{+}),$$

$$-i=(-i)2+i, \quad (i \in <\cdot|_{2}^{+}),$$

and for p.s.v. {0, 1} on mod(1-i) is one modular number: -i, since $-i=(-i)(1-i)+1$ $(1 \in <\cdot|_{1-i}^{+})$.

Case 2. For p.s.n.v. on mod(c+di), where c<0, modular numbers there does not exist.

Actually/really, let $\alpha+\beta i=1((-c'-1)+di)$, where $\alpha+\beta i \in <\cdot|_{p}^{+}$ and $c=-c'$ $(c'>0)$, then

$$\|\alpha+\beta i\| \geqslant \|(-c'-1)+di\| = (c'+1)^2+d^2 > \|p\|,$$

which leads to the contradiction.

For p.s.v. {0, 1} on mod(-1+i) modular numbers.

Case of 3. For p.s.n.v. on mod (c+di) modular numbers there does not exist. Actually/really, let $\alpha+\beta i=q((c-1)+di)$, where $\alpha+\beta i \in <\cdot|_{p}^{-}$, then

$$\|\alpha+\beta i\| \geqslant (c-1)^2+d^2=\frac{c^2+d^2}{2}+\left(\frac{c^2+d^2}{2}-2c+1\right).$$

Consequently, for all those cases when

$$\frac{c^2+d^2}{2}-2c+1>0, \tag{4.3}$$

initial assumption proves to be inaccurate, since otherwise we come to the inaccurate conclusion

$$\|\alpha+\beta i\| > \frac{\|p\|}{2}.$$

The moduli/modules, not included by inequality (4.3), are such: 2+-i, 3, 3+-i. By direct testing we are convinced, that for p.s.a.n.v. on these moduli/modules there does not exist modular numbers.

Properties of the symmetry of p.s.a.n.v. One of advantages of p.s.a.n.v. on any modulus/module is the absence of modular numbers. Furthermore, p.s.a.n.v. possess the series/row of other advantages before other types of p.s.v. in view of their properties symmetries. These properties are expressed by formulas.

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Theorem 4.2. If ts.k.ch. p=c+di odd, then

$$1.\ <i(a+bi)\,|_{p}^{-} = i\cdot<a+bi\,|_{p}^{-},$$

$$2.\ \overline{<a+bi\,|_{p}^{-}} = <\overline{a+bi}\,|_{p}^{-},$$

$$3.\ \underline{<a+bi\,|_{p}^{-}} = <\underline{a+bi}\,|_{p}^{-}.$$

Here $\overline{a+bi}$ and $\underline{a+bi}$ indicate

$$\overline{a+bi}=a-bi;\ \underline{a+bi}=-a+bi.$$

Proof. In view of the basic formula of deductions (3.1.5) we have

$$i<a+bi\,|_{p}^{-} = \frac{(-\,|\,bc-ad\,|_{\|p\|}^{-}+i\,|\,ac+bd\,|_{\|p\|}^{-})\,p}{\|p\|}$$

and

$$< i\,(a+bi)^{-}_{p} = \frac{\left(|-bc+ad|^{-}_{\|p\|} + i\,|ac+bd|^{-}_{\|p\|}\right) p}{\|p\|}.$$

but when $\|p\| \equiv 1 \pmod 2$

$$|-bc+ad|^{-}_{\|p\|} = -\,|bc-ad|^{-}_{\|p\|},$$

therefore $< i\,(a+bi)\,|^{-}_{p} = i < a+bi\,|^{-}_{p}$.

Further, since

$$\overline{< a+bi\,|}_{p} = \frac{\left(|ac+bd|^{-}_{\|p\|} - i\,|bc-ad|^{-}_{\|p\|}\right)(c-di)}{\|p\|}$$

and

$$< a-bi\,|^{-}_{p} = \frac{\left(|ac+bd|^{-}_{\|p\|} + i\,|-bc+ad|^{-}_{\|p\|}\right)(c-di)}{\|p\|},$$

that

$$\overline{< a+bi\,|}_{p} = < \overline{a+bi}\,|^{-}_{p}.$$

Finally, since $\underline{z} = -\overline{z}$, then

$$\underline{< a+bi\,|}_{p} = -\overline{< a+bi\,|}_{p} = -< \overline{a+bi}\,|^{-}_{p} =$$

$$= < -(\overline{a+bi})\,|^{-}_{p} = \underline{< a+bi\,|}^{-}_{p}.$$

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Observation 1. For ts.k.ch., the generatrices p.s.a.n.v. on the real modulus/module p, the enumerated properties take the form

$$i < a+bi\,|_p = < -b+ai\,|_p,$$

$$< \overline{a+bi\,|}_p = < \overline{a+bi}\,|_p,$$

$$< \underline{a+bi\,|}_p = < a+bi\,|_p.$$

Observation 2. In view of theorem 2.2.2, property in their arithmetic sense they can be spread to arbitrary p.s.v. over mod p, $(\|p\| \equiv 1 (\mathrm{mod}\ 2))$.

Example (Fig. 27 and 28). mod(3±6i). Let us write out the group of deductions p.s.a.n.v. on mod(3+6i), arranged/located in the first quadrant,

$$1,\ 2,\ 3,\ 1+i,\ 2+i,\ 3+i,\ 4+i,\ 1+2i,\ 2+2i,\ 3+2i,\ 1+4i.$$

By the force of the first property all remaining nonzero deductions can be obtained by the additional multiplication of this group of deductions to the consecutive degrees of the imaginary unit

$$i,\ 2i,\ 3i,\ -1+i,\ -1+2i,\ -1+3i,\ -1+4i,\ -2+i,\ -2+2i,\ -2+3i,\ -4+i;$$

$$-1,\ -2,\ -3,\ -1-i,\ -2-i,\ -3-i,\ -4-i,\ -1-2i,\ -2-2i,\ -3-2i,$$
$$-1-4i;$$

$$-i,\ -2i,\ -3i,\ 1-i,\ 1-2i,\ 1-3i,\ 1-4i,\ 2-i,\ 2-2i,\ 2-3i,\ 4-i.$$

The second and third properties are illustrated by the following examples:

$$-1+4i \in \langle \cdot |_{\bar{3}+6i}, \qquad -1+4i \in \langle \cdot |_{\bar{3}+6i}$$

$$-1-4i \in \langle \cdot |_{\bar{3}-6i}, \qquad 1+4i \in \langle \cdot |_{\bar{3}-6i}$$

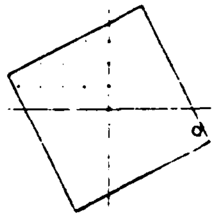

Fig. 27. mod (3+6i).

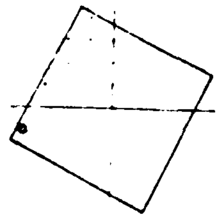

Fig. 28.

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For ts.k.ch. p=c+di such, that $\|p\| \equiv 0 \pmod 2$ (i.e. for the semieven and the the even ts.k.ch.) the property of the symmetry of deductions p.s.a.n.v. is broken in connection with the presence on the sides of the squares p.s.a.n.v. of the integer points, from which in p.s.a.n.v. are included only the points, arranged/located on any two adjacent sides of square.

§ 5. Analysis of the tables of modular operations.

The tables of modular operations compose the basis of any numeration system. Under the table of modular operation is understood the two-input table, salient on two input operands (deductions) the result of modular operation (addition, multiplication on the preset

modulus/module).

As an example are given the tables of modular addition and multiplication on mod(3+2i), where the operands (deductions) are preset in the class of p.s.n.v.

The analysis of this type of tables implies:

1) the determination of the character of overflow with the execution of modular operation;

2) the development/detection of the properties of the symmetry of the tables of modular operations. There is greatest interest for the machine arithmetic in the determination of the character of overflow with the addition.

Overflow with the addition of the elements/cells of p.s.n.v. We will use the identity, valid on many elements/cells of p.s.v. If

$\alpha+\beta i \in \langle\cdot|_p, (p=c+di)$, then

$$\alpha+\beta i=((\alpha c+\beta d)+i(\beta c-\alpha d))\frac{c+di}{\|p\|}.$$ Let

$\gamma_1=\alpha_1+\beta_1 i$ $\gamma_2=\alpha_2+\beta_2 i \in \langle\cdot|_p^+$, then

$$\gamma_1+\gamma_2=\{(\alpha_1 c+\beta_1 d)+(\alpha_2 c+\beta_2 d)+i((\beta_1 c-\alpha_1 d)+$$
$$+(\beta_2 c-\alpha_2 d))\}\frac{c+di}{\|p\|}.$$

Since for p.s.n.v. it is carried out

$$0\leqslant\alpha_1 c+\beta_1 d<\|p\|,\ 0\leqslant\alpha_2 c+\beta_2 d<\|p\|,$$
$$0\leqslant\beta_1 c-\alpha_1 d<\|p\|,\ 0\leqslant\beta_2 c-\alpha_2 d<\|p\|.$$

then

$$(\alpha_1 c + \beta_1 d) + (\alpha_2 c + \beta_2 d) = (\alpha_1 + \alpha_2) c + (\beta_1 + \beta_2) d =$$
$$= |(\alpha_1 + \alpha_2) c + (\beta_1 + \beta_2) d|^{+}_{\|p\|} + \gamma_1 \|p\|,$$

where

$$\gamma_1 = \left[\frac{(\alpha_1 + \alpha_2) c + (\beta_1 + \beta_2) d}{\|p\|}\right]^{+} = \begin{cases} 0 \\ 1 \end{cases}$$

(symbol $[x]^{+}$ designated whole syllable x, equal to the greatest integer, which does not exceed x).

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Table of modular summation over mod(3+2i).

$+$	0	$-1+3i$	$3i$	$1+3i$	$2+3i$	i	$1+i$	$4i$	$1+4i$	$-1+2i$	$2i$	$1+2i$	$2+2i$
0	0	$-1+3i$	$3i$	$1+3i$	$2+3i$	i	$1+i$	$4i$	$1+4i$	$-1+2i$	$2i$	$1+2i$	$2+2i$
$-1+3i$	$-1+3i$	$3i$	$1+3i$	$2+3i$	i	$1+i$	$4i$	$1+4i$	$-1+2i$	$2i$	$1+2i$	$2+2i$	0
$3i$	$3i$	$1+3i$	$2+3i$	i	$1+i$	$4i$	$1+4i$	$-1+2i$	$2i$	$1+2i$	$2+2i$	0	$-1+3i$
$1+3i$	$1+3i$	$2+3i$	i	$1+i$	$4i$	$1+4i$	$-1+2i$	$2i$	$1+2i$	$2+2i$	0	$-1+3i$	$3i$
$2+3i$	$2+3i$	i	$1+i$	$4i$	$1+4i$	$-1+2i$	$2i$	$1+2i$	$2+2i$	0	$-1+3i$	$3i$	$1+3i$
i	i	$1+i$	$4i$	$1+4i$	$-1+2i$	$2i$	$1+2i$	$2+2i$	0	$-1+3i$	$3i$	$1+3i$	$2+3i$
$1+i$	$1+i$	$4i$	$1+4i$	$-1+2i$	$2i$	$1+2i$	$2+2i$	0	$-1+3i$	$3i$	$1+3i$	$2+3i$	i
$4i$	$4i$	$1+4i$	$-1+2i$	$2i$	$1+2i$	$2+2i$	0	$-1+3i$	$3i$	$1+3i$	$2+3i$	i	$1+i$
$1+4i$	$1+4i$	$-1+2i$	$2i$	$1+2i$	$2+2i$	0	$-1+3i$	$3i$	$1+3i$	$2+3i$	i	$1+i$	$4i$
$-1+2i$	$-1+2i$	$2i$	$1+2i$	$2+2i$	0	$-1+3i$	$3i$	$1+3i$	$2+3i$	i	$1+i$	$4i$	$1+4i$
$2i$	$2i$	$1+2i$	$2+2i$	0	$-1+3i$	$3i$	$1+3i$	$2+3i$	i	$1+i$	$4i$	$1+4i$	$-1+2i$
$1+2i$	$1+2i$	$2+2i$	0	$-1+3i$	$3i$	$1+3i$	$2+3i$	i	$1+i$	$4i$	$1+4i$	$-1+2i$	$2i$
$2+2i$	$2+2i$	0	$-1+3i$	$3i$	$1+3i$	$2+3i$	i	$1+i$	$4i$	$1+4i$	$-1+2i$	$2i$	$1+2i$

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Table of modular multiplication on mod(3+2i).

$\times$	0	$-1+3i$	$3i$	$1+3i$	$2+3i$	i	$1+i$	$4i$	$1+4i$	$-1+2i$	$2i$	$1+2i$	$2+2i$
0	0	0	0	0	0	0	0	0	0	0	0	0	0
$-1+3i$	0	$-1+3i$	$3i$	$1+3i$	$2+3i$	i	$1+i$	$4i$	$1+4i$	$-1+2i$	$2i$	$1+2i$	$2+2i$
$3i$	0	$3i$	$2+3i$	$1+i$	$1+4i$	$2i$	$2+2i$	$-1+3i$	$1+3i$	i	$4i$	$-1+2i$	$1+2i$
$1+3i$	0	$1+3i$	$1+i$	$-1+2i$	$2+2i$	$3i$	i	$1+4i$	$1+2i$	$-1+3i$	$2+3i$	$4i$	$2i$
$2+3i$	0	$2+3i$	$1+4i$	$2+2i$	$1+3i$	$4i$	$1+2i$	$3i$	$2+i$	$2i$	$-1+3i$	i	$-1+2i$
i	0	i	$2i$	$3i$	$4i$	$2+2i$	$2+3i$	$-1+2i$	$-1+3i$	$1+i$	$1+2i$	$1+3i$	$1+4i$
$1+i$	0	$1+i$	$2+2i$	i	$1+2i$	$2+3i$	$2i$	$1+3i$	$-1+2i$	$3i$	$1+4i$	$-1+3i$	$4i$
$4i$	0	$4i$	$-1+3i$	$1+4i$	$3i$	$-1+2i$	$1+3i$	$2i$	$2+3i$	$1+2i$	i	$2+2i$	$1+i$
$1+4i$	0	$1+4i$	$1+3i$	$1+2i$	$2+i$	$-1+3i$	$-1+2i$	$2+3i$	$2+2i$	$4i$	$3i$	$2i$	i
$-1+2i$	0	$-1+2i$	i	$-1+3i$	$2i$	$1+i$	$3i$	$1+2i$	$4i$	$1+3i$	$2+2i$	$1+4i$	$2+3i$
$2i$	0	$2i$	$4i$	$2+3i$	$-1+3i$	$1+2i$	$1+4i$	i	$3i$	$2+2i$	$-1+2i$	$1+i$	$1+3i$
$1+2i$	0	$1+2i$	$-1+2i$	$4i$	i	$1+3i$	$-1+3i$	$2+2i$	$2i$	$1+4i$	$1+i$	$2+3i$	$3i$
$2+2i$	0	$2+2i$	$1+2i$	$2i$	$-1+2i$	$1+4i$	$4i$	$1+i$	i	$2+3i$	$1+3i$	$3i$	$-1+3i$

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Is analogous $(\beta_1 c-\alpha_1 d)+)(\beta_2 c-\alpha_2 d)=|(\beta_1+\beta_2)\ c-(\alpha_1+\alpha_2 d)$
$|^{+}_{\|p\|}+\delta\|p\|$, where

$$\delta=\left[\frac{(\beta_1+\beta_2)\,c-(\alpha_1+\alpha_2)\,d}{\|p\|}\right]^{+}=\begin{cases}0\\1\end{cases}.$$

Thus,

$$(\alpha_1+\beta_1 i)+(\alpha_2+\beta_2 i)=(\gamma_1+i\delta)\ (c+di)+(\ |\ (\alpha_1+\alpha_2)\,c+$$
$$+(\beta_1+\beta_2)\,d\ |^{+}_{p},+i\ |\ (\beta_1+\beta_2)\,c-\ (\alpha_1+\alpha_2)\,d\ |^{+}_{\|p\|})\ \frac{c-di}{\|p\|}\ .$$

Second term of the right side of the latter/last equality, in view of basic formula for deductions (3.1.5), is the modular sum of deductions $\alpha_1+\beta_1 i$ and $\alpha_2+\beta_2 i$. Thus, possible overflows with the addition of two deductions of p.s.n.v. take the form

$$\pi(\zeta_1+\zeta_2)=\eta+\delta i,$$

where

$$\eta,\ \delta=\begin{cases}0\\1\end{cases}.$$

In other words, transfer π with the addition of two composite deductions ζ_1 and ζ_2 on modulus/module p=c+di is realized by one of the numbers of the set: 0, 1, i, 1+i. Hence, in particular, it follows that with the addition a control of departure beyond the range, preset p.s.n.v., can be carried out by means of modulus/module 2, since $<\cdot|_2^+=\{0,1,i,1+i\}$. The character of overflow is here such: it is not possible to indicate such composite modulus/module p=c+di (c, d≠0), p.s.n.v. which would contain in themselves all numbers 0, 1, i, 1+i.

Consequently, with addition of two ts.k.ch., preset by the positional code

$$\alpha_0+\alpha_1 p+\alpha_2 p^2+\ldots+\alpha_n p^n,$$

where

$$\alpha_k \in <\cdot|_p^+\quad (k=0,1,2,\ldots,n),$$

the overflow, which appears in this digit, can affect not only the directly next decade.

For an example let us give the table of overflows with the addition of the smallest deductions on mod(3+2i).

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Table of overflows with the summation over mod(3+2i).

$\pi(\xi_1+\xi_2)$	0	$-1+3i$	$3i$	$1+3i$	$2+3i$	i	$1+i$	$4i$	$1+4i$	$-1+2i$	$2i$	$1+2i$	$2+2i$
0	0	0	0	0	0	0	0	0	0	0	0	0	0
$-1+3i$	0	i	i	i	$1+i$	i	0	i	$1+i$	i	i	i	$1+i$
$3i$	0	i	i	$1+i$	$1+i$	0	0	$1+i$	$1+i$	i	i	$1+i$	1
$1+3i$	0	i	$1+i$	$1+i$	1	0	1	$1+i$	$1+i$	i	$1+i$	1	1
$2+3i$	0	$1+i$	$1+i$	1	1	1	1	$1+i$	$1+i$	$1+i$	1	1	1
i	0	i	0	0	1	0	0	i	$1+i$	0	0	0	0
$1+i$	0	0	0	1	1	0	0	$1+i$	1	0	0	0	1
$4i$	0	i	$1+i$	$1+i$	$1+i$	i	$1+i$	$1+i$	$1+i$	i	i	$1+i$	$1+i$
$1+4i$	0	$1+i$	$1+i$	$1+i$	$1+i$	$1+i$	1	$1+i$	$1+i$	i	$1+i$	$1+i$	1
$-1+2i$	0	i	i	i	$1+i$	0	0	i	i	i	i	0	0
$2i$	0	i	i	$1+i$	1	0	0	i	$1+i$	i	0	0	1
$1+2i$	0	i	$1+i$	1	1	0	0	$1+i$	$1+i$	0	0	1	1
$2+2i$	0	$1+i$	1	1	1	0	1	$1+i$	1	0	1	1	1

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Overflow with the addition of the elements/cells of p.s.a.n.v.

Let $\alpha_1+\beta_1 i$, $\alpha_2+\beta_2 i \in \langle \cdot \mid_p$, $(p=c+di)$, then

$$(\alpha_1+\beta_1 i)+(\alpha_2+\beta_2 i)+\{(\alpha_1 c+\beta_1 d)+(\alpha_2 c+\beta_2 d)+$$
$$+i((\beta_1 c-\alpha_1 d)+(\beta_2 c-\alpha_2 d))\}\frac{c+di}{\|p\|}.$$

Since for p.s.a.n.v. it is carried out

$$-\frac{\|p\|}{2}\leqslant \alpha_1 c+\beta_1 d<\frac{\|p\|}{2};\ -\frac{\|p\|}{2}\leqslant \alpha_2 c+\beta_2 d<\frac{\|p\|}{2},$$
$$-\frac{\|p\|}{2}\leqslant \beta_1 c-\alpha_1 d<\frac{\|p\|}{2};\ -\frac{\|p\|}{2}\leqslant \beta_2 c-\alpha_2 d<\frac{\|p\|}{2},$$

then

$$(\alpha_1 c + \beta_1 d) + (\alpha_2 c + \beta_2 d) = \| (\alpha_1 + \alpha_2) c +$$
$$+ (\beta_1 + \beta_2) d \|^-_{p} + \eta \| p \|,$$

where

$$\eta = \left[\frac{(\alpha_1 + \alpha_2) c + (\beta_1 + \beta_2) d}{\| p \|} \right]^- = \begin{cases} -1 \\ 0 \\ +1 \end{cases}$$

(by symbol $[x]^-$ markedly integer, near to x).

Is analogous $(\beta_1 c-\alpha_1 d)+(\beta_2 c-\alpha_2 d)=\|(\beta_1+\beta_2)$ c- $(\alpha_1+\alpha_2)$with d) $\|^-_{p} + \delta \| p \|$, where

$$\delta = \left[\frac{(\beta_1 + \beta_2) c - (\alpha_1 + \alpha_2) d}{\| p \|} \right]^- = \begin{cases} -1 \\ 0. \\ +1 \end{cases}$$

Hence, it is similar to the previous case, we consist that the possible overflows $\pi(\zeta_1+\zeta_2)$ with the addition of two deductions ζ_1 and ζ_2 p.s.a.n.v. take the form

$$\pi(\zeta_1 + \zeta_2) = \eta + \delta i,$$

where

$$\eta, \delta = \begin{cases} -1 \\ 0. \\ +1 \end{cases}$$

i.e. transfer w with the addition of two composite deductions p.s.a.n.v. on modulus/module p=c+di is realized by one of the numbers of set 0, 1, 1+i, i, -1+i, -1, -1-i, -i, 1-i.

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Tabla of modular addition p.s.a.n.v. on mod(3+2i).

$+$	0	1	2	$-2i$	$-1+i$	i	$1+i$	$-1-i$	$-i$	$1-i$	$2i$	-2	-1
0	0	1	2	$-2i$	$-1+i$	i	$1+i$	$-1-i$	$-i$	$1-i$	$2i$	-2	-1
1	1	2	$-2i$	$-1+i$	i	$1+i$	$-1-i$	$-i$	$1-i$	$2i$	-2	-1	0
2	2	$-2i$	$-1+i$	i	$1+i$	$-1-i$	$-i$	$1-i$	$2i$	-2	-1	0	1
$-2i$	$-2i$	$-1+i$	i	$1+i$	$-1-i$	$-i$	$1-i$	$2i$	-2	-1	0	1	2
$-1+i$	$-1+i$	i	$1+i$	$-1-i$	$-i$	$1-i$	$2i$	-2	-1	0	1	2	$-2i$
i	i	$1+i$	$-1-i$	$-i$	$1-i$	$2i$	-2	-1	0	1	2	$-2i$	$-1+i$
$1+i$	$1+i$	$-1-i$	$-i$	$1-i$	$2i$	-2	-1	0	1	2	$-2i$	$-1+i$	i
$-1-i$	$-1-i$	$-i$	$1-i$	$2i$	-2	-1	0	1	2	$-2i$	$-1+i$	i	$1+i$
$-i$	$-i$	$1-i$	$2i$	-2	-1	0	1	2	$-2i$	$-1+i$	i	$1+i$	$-1-i$
$1-i$	$1-i$	$2i$	-2	-1	0	1	2	$-2i$	$-1+i$	i	$1+i$	$-1-i$	$-i$
$2i$	$2i$	-2	-1	0	1	2	$-2i$	$-1+i$	i	$1+i$	$-1-i$	$-i$	$1-i$
-2	-2	-1	0	1	2	$-2i$	$-1+i$	i	$1+i$	$-1-i$	$-i$	$1-i$	$2i$
-1	-1	0	1	2	$-2i$	$-1+i$	i	$1+i$	$-1-i$	$-i$	$1-i$	$2i$	-2

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Consequently, with the addition a control of output/yield on the range, preset p.s.a.n.v., can be realized by means of modulus/module 3, since

$$\langle \cdot \mid_3^- = \{0,\ 1,\ 1+i,\ i,\ -1+i,\ -1,\ -1-i,\ -i,\ 1-i\}.$$

In contrast to p.s.n.v. the character of overflow is such: p.s.a.n.v. on any modulus/module $p\,(\|p\|>2)$ it contains as the

deductions of a number of form 0, 1, 1+i, i, -1+i, -1, -1-i, -i, 1-i.

As an example are given the tables of modular addition and overflows p.s.a.n.v. on mod(3+2i) and mod(1-2i).

The table of overflows with the addition of a.n.v. on mod(3+2i).

$\pi(\xi_1-\xi_2)$	0	1	2	$-2i$	$-1+i$	i	$1+i$	$-1-i$	$-i$	$1-i$	$2i$	-2	-1
0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	0	1	$-i$	0	0	1	0	0	$-i$	1	0	0
2	0	1	$1-i$	$-i$	0	1	1	0	$-i$	$1-i$	1	0	0
$-2i$	0	$-i$	$-i$	$-1-i$	0	0	0	$-1-i$	$-i$	$-i$	0	-1	-1
$-1-i$	0	0	0	0	i	i	0	0	0	0	i	$-1+i$	i
i	0	0	1	0	i	0	1	0	0	0	i	i	0
$1-i$	0	1	1	0	0	1	1	0	0	0	$1+i$	0	0
$-1-i$	0	0	0	$-1-i$	0	0	0	-1	-1	0	0	-1	-1
$-i$	0	0	$-i$	$-i$	0	0	0	-1	0	0	0	-1	0
$1-i$	0	$-i$	$1-i$	$-i$	0	0	0	0	0	$-i$	0	0	0
$2i$	0	1	1	0	i	i	$1+i$	0	0	0	$1-i$	i	i
-2	0	0	0	-1	$-1+i$	i	0	-1	-1	0	i	$-1+i$	-1
-1	0	0	0	-1	i	0	0	-1	0	0	i	-1	0

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Tables show that the overflows with the addition a.n.v. on mod(1-2i) are again elements/cells p.s.a.n.v. on mod(1-2i).

Overflows with the addition of real remainders on the composite modulus/modula p=c+di ((c, d)=1).

Case of p.s.n.n. v. $\langle\cdot|_p = |\cdot|^{-}_p$.

Let $x_1, x_2 \in |\cdot|^{-}_{\|p}$, then

$$x_1 + x_2 = | x_1 + x_2 |^{+}_{p} + \eta p.$$

where

$$\eta = \left[\frac{x_1 - x_2}{p}\right]^{-} = \begin{cases} 0 \\ 1 \end{cases}.$$

Since $\|p\| = \bar{p} \cdot p$, the overflow in this case accepts form $\eta(c-di)$.

Let it be $c>0$, then ts.k.ch. $\bar{p}$ it is possible to present in the form $\bar{p}=2c-p$.

In this case $2c \in |\cdot|^{-}_{\|p\|}$, since $2c<c^2+d^2$, since $c \neq d$ on the strength of the fact that $(c, d) = 1$. Consequently, the character of overflow with the addition of two real deductions is such:

$$x_1 + x_2 = | x_1 + x_2 |^{+}_{\|p} + \pi p,$$

where

$$\pi(x_1 + x_2) = \begin{cases} 0 \\ 2c - p \end{cases}.$$

The table of modular addition p.s.a.n.v. on mod(1-2i).

$+$	1	i	-1	$-i$
1	$-i$	-1	0	i
i	-1	1	$-i$	0
-1	0	$-i$	i	1
$-i$	i	0	1	-1

Table of ovefilling on mod(1-2i).

π^{+}	1	i	-1	$-i$
1	i	i	0	1
i	i	-1	-1	0
-1	0	-1	$-i$	$-i$
$-i$	1	0	$-i$	1

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Number -1 in this case is a modular number, since

$$-1 = (\|p\| - 1) + (\|p\| - 2c)\,p + (1 - \bar{p})\,p^2,$$

but

$$1 - \bar{p} = \|p\| - (2c - 1) + (1 - \bar{p})\,p.$$

All this substantially complicates the processes of transfer.

Case of p.s.a.n.v. $<\cdot|_p = |\cdot|^{-1}_{\|p\|}$.

Let $\alpha_1, \alpha_2 \in |\cdot|^-_{\|p\|}$, then

$$\alpha_1+\alpha_2=|\alpha_1+\alpha_2|^-_{\|p\|}+\pi\|p\|,$$

where

$$\pi(\alpha_1+\alpha_2)=\begin{cases}-1\\ 0.\\ 1\end{cases}$$

Let, for example, $c<\frac{\|p\|}{4}$, then ts. k.ch. $\overline{p}$ be is represented in the form $\overline{p}$=2c-p.

Since $2c\in|\cdot|^-_{\|p\|}, -1\in|\cdot|^-_{\|p\|}$, that the value of overflow in this case is described by the "two-place" positional code.

Overflows with the multiplication of the elements/cells of p.s.v. let us consider the case of p.s.a.n.v. Let $\zeta_1=\alpha_1+\beta_1 i$, $\zeta_2=\alpha_2+\beta_2 i\in<\cdot|^-_p(p=c+di)$, then

$$(\alpha_1+\beta_1 i)(\alpha_2+\beta_2 i)=(\alpha_1+\beta_1 i)(\alpha_2 c+\beta_2 d)+$$
$$+i(\beta_2 c-\alpha_2 d))\frac{c+di}{\|p\|}=\left\{\left[\frac{\alpha_1(\alpha_2 c+\beta_2 d)-\beta_1(\beta_2 c-\alpha_2 d)}{\|p\|}\right]^-+\right.$$
$$\left.+i\left[\frac{\alpha_1(\beta_2 c-\alpha_2 d)+\beta_1(\alpha_2 c+\beta_2 d)}{\|p\|}\right]^-\right\}(c+di)+(|\alpha_1(\alpha_2 c+\beta_2 d)-$$
$$-\beta_1(\beta_2 c-\alpha_2 d)|^-_{\|p\|}+i|\alpha_1(\beta_2 c-\alpha_2 d)+$$
$$+\beta_1(\alpha_2 c+\beta_2 d)|_{\|p\|})\frac{c+di}{\|p\|}.$$

Since

$$\alpha_1(\alpha_2 c+\beta_2 d)-\beta_1(\beta_2 c-\alpha_2 d)=(\alpha_1\alpha_2-\beta_1\beta_2)c+$$
$$+(\alpha_1\beta_2+\alpha_2\beta_1)d,$$
$$\alpha_1(\beta_2 c-\alpha_2 d)+\beta_1(\alpha_2 c+\beta_2 d)=(\alpha_1\beta_2+\alpha_2\beta_1)c-$$
$$-(\alpha_1\alpha_2-\beta_1\beta_2)d.$$

that the value of overflow $\pi(\zeta_1 \cdot \zeta_2)$ with the multiplication a.n.v. ζ_1 and ζ_2 is calculated according to the formula

$$\pi(\zeta_1, \zeta_2) = [\varkappa_1]^- + i[\varkappa_2]^-,$$

where

$$\varkappa_1 = \frac{\alpha_1(\alpha_2 c + \beta_2 d) - \beta_1(\beta_2 c - \alpha_2 d)}{\| p \|}; \quad \varkappa_2 = \frac{\alpha_1(\beta_2 c - \alpha_2 d) + \beta_1(\alpha_2 c + \beta_2 d)}{\| p \|}.$$

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Table of modular multiplication p.s.a.n.v. on mod (2+3i).

	0	1	2	-1	-2	i	$-i$	$2i$	$-2i$	$1+i$	$1-i$	$-1+i$	$-1-i$
0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	1	2	-1	-2	i	$-i$	$2i$	$-2i$	$1+i$	$1-i$	$-1+i$	$-1-i$
2	0	2	$-1-i$	-2	$1+i$	$2i$	$-2i$	$1-i$	$-1+i$	$-i$	-1	1	i
-1	0	-1	-2	1	2	$-i$	i	$-2i$	$+2i$	$-1-i$	$-1+i$	$1-i$	$1+i$
-2	0	-2	$1+i$	2	$-1-i$	$-2i$	$2i$	$-1+i$	$1-i$	i	1	-1	$-i$
i	0	i	$2i$	$-i$	$-2i$	-1	1	-2	2	$-1-i$	$1+i$	$-1-i$	$1-i$
$-i$	0	$-i$	$-2i$	$+i$	$2i$	1	-1	2	-2	$1-i$	$-1-i$	$1+i$	$-1+i$
$2i$	0	$2i$	$1-i$	$-2i$	$-1+i$	-2	2	$1+i$	$-1-i$	1	$-i$	i	-1
$-2i$	0	$-2i$	$-1+i$	$2i$	$1-i$	2	-2	$-1-i$	$1+i$	-1	i	$-i$	1
$1+i$	0	$1+i$	$-i$	$-1-i$	i	$-1+i$	$1-i$	1	-1	$2i$	2	-2	$-2i$
$1-i$	0	$1-i$	-1	$-1+i$	1	$1+i$	$-1-i$	$-i$	i	2	$-2i$	$2i$	-2
$-1+i$	0	$-1+i$	1	$1-i$	-1	$-1-i$	$1+i$	i	$-i$	-2	$2i$	$-2i$	2
$-1-i$	0	$-1-i$	i	$1+i$	$-i$	$1-i$	$-1+i$	-1	1	$-2i$	-2	2	$2i$

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Certain representation about the value of overflow give the evaluations

$$|x_1| \leqslant \frac{|\alpha_1| + |\beta_1|}{2}, \quad |x_2| \leqslant \frac{|\alpha_1| + |\beta_1|}{2}$$

or

$$|x_1 + ix_2| \leqslant \sqrt{2}\,|\alpha_1 + i\beta_1|,$$

where $\alpha_1 + i\beta_1$ - smaller in the absolute value cofactor.

The table of overflows with the modular multiplication on mod(2+3i).

x_x	0	1	2	-1	-2	i	$-i$	$2i$	$-2i$	$1+i$	$1-i$	$-1+i$	$-1-i$
0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	0	0	0	0	0	0
2	0	0	$1-i$	0	$-1+i$	0	0	$1+i$	$-1-i$	1	$-i$	i	$-i$
-1	0	0	0	0	0	0	0	0	0	0	0	0	0
-2	0	0	$-1+i$	0	$1+i$	0	0	$-1-i$	$-1+i$	-1	i	$-i$	1
i	0	0	0	0	0	0	0	0	0	0	0	0	0
$-i$	0	0	0	0	0	0	0	0	0	0	0	0	0
$2i$	0	0	$1+i$	0	$-1-i$	0	0	$-1+i$	$1-i$	i	1	-1	$-i$
$-2i$	0	0	$-1-i$	0	$1+i$	0	0	$1-i$	$-1+i$	$-i$	-1	1	i
$1+i$	0	0	1	0	-1	0	0	i	$-i$	0	0	0	0
$1-i$	0	0	$-i$	0	i	0	0	1	-1	0	0	0	0
$-1+i$	0	0	i	0	$-i$	0	0	-1	1	0	0	0	0
$-1-i$	0	0	-1	0	1	0	0	$-i$	i	0	0	0	0

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The evaluations indicated are the immediate consequence of the inequalities

$$|\alpha_2 c+\beta_2 d| \leqslant \frac{\|p\|}{2}, \quad |\beta_2 c-\alpha_2 d| \leqslant \frac{\|p\|}{2}.$$

Multiplication table on mod(1-2i) is convenient fact that the overflows with the multiplication of any two deductions on the modulus/module indicated are equal to 0.

Let us switch over to a question about the methods of the

reduction of modular tables. With an increase in the norm of modulus/module the tables of modular operations become bulky, which, naturally, leads to an increase in the equipment and it affects triggering time of the device/equipment, which realizes modular operation. In connection with this there is the great practical interest in the question about the abridgement of table of modular operations, which in turn, is connected with questions of the special coding of deductions.

The in practice satisfactory solution of this question can be obtained, using planar symmetry of p.s.a.n.v.

Let us consider at first odd moduli/modules $(\|p\| \equiv 1 \pmod 2)$. In this case the set of all deductions on mod p can be decomposed on $\frac{\|p\|-1}{4}$ the groups of the associated deductions.

Choosing on one representative of each group of the associated deductions and labeling them in the desired exponent 1, 2, 3, ..., $\frac{\|p\|-1}{4}$, we will obtain the sequence which let us name the sequence of the deductions of rank 0. The sequence of deductions, obtained by additional multiplication on i^{λ} each element/cell of the sequence of the deductions of rank 0 and that arranged/located in the appropriate order, we will call consequence of the deductions of rank λ. The number of deduction in this case we will call the mantissa of

deduction.

Thus, planar symmetry p.s.a.n.v. makes it possible to introduce the general-purpose coding of deductions for all p.s.v. on the odd moduli/modules.

Any deduction $\alpha+\beta i \in <\cdot|_p^-$ $(\|p\| \equiv 1 \pmod 2)$ is unambiguously represented in the form $\alpha+\beta i=\lambda.\ n$, where λ - rank, while n - mantissa of deduction (point it is used as separating symbol).

The table of the modular multiplication of deductions p.s.a.n.v. on mod(1-2i).

$\times$	1	i	-1	$-i$
1	1	i	-1	$-i$
i	i	-1	$-i$	1
-1	-1	$-i$	1	i
$-i$	$-i$	1	i	-1

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Rank λ can take values of 0, 1, 2, 3, i.e., value from p.s.n.v. on mod 4, and mantissa is coded by numbers 1, 2, 3, ..., $\frac{p-1}{4}$. Modular operations respectively take the form

$$< (\lambda_1 . n_1) \cdot (\lambda_2 . n_2) \mid_p^- = (\mid \lambda_1 + \lambda_2 \mid_4^+ , 1) \times$$
$$\times < (0. n_1) \times (0. n_2) \mid_{\| p \|}^- , \qquad (5.1)$$
$$< \lambda_1 . n_1 + \lambda_2 . n_2 \mid_p^- = (\lambda_1 . 1) \cdot < 0. n_1 +$$
$$+ \mid \lambda_2 - \lambda_1 \mid_4^+ . n_2 \mid_p^- . \qquad (5.2)$$

Here it is thought that deduction 1 answers the mantissa, equal to 1. Thus, with the multiplication of deductions the ranks of deductions store/add up on mod 4, mantissas are multiplied in accordance with the abbreviated/reduced modular table, and during the addition is present the operation of the standardization of first term (analog of the operation of the matching of exponents of the positional coding of deductions).

Because of this method of organizing the modular operations of multiplication table are reduced 16 times (in this case still remains the possibility to shorten multiplication tables due to the commutation of the operation of multiplication), and the tables of addition - 4 times.

Example. P.s.a.n.v. cn mod (3+2i) let us code in accordance with the table

$< \cdot \mid_{\overline{3 \cdot 2i}}$	0	1	2	$1+i$	i	$2i$	$-1+i$	-1	-2	$-1-i$	$-i$	$-2i$	$1-i$
Код (1)	0.0	0.1	0.2	0.3	1.1	1.2	1.3	2.1	2.2	2.3	3.1	3.2	3.3

Key: (1). Code.

Then the abbreviated/reduced tables of modular multiplication and addition take the following form.

The table of modular multiplication on mod(3+2i) [1].

	00	01	02	03
00	0	0	0	0
01	0	01	02	03
02	0	02	13	21
03	0	03	21	12

FOOTNOTE [1]. Here and below in the tables for the convenience are omitted separating points. ENDFOOTNOTE.

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It is not difficult to see that to the tables of overflow also extends principle examined above of abridgement of table.

Actually/really, from formulas (5.1) and (5.2) it follows that it suffices to assign the table of overflows

$$\pi((0, n_1)\cdot(0, n_2))_p$$

$$\pi((0, n_1) - (0, n_2))_p$$

in order to know overflows in any other cases. The overflows, which appear in the rejected/thrown part of the tables, will differ from those overflows which are considered in the abbreviated/reduced table by the dividers/denominators of unity:

$$\pi((\lambda_1, n_1)\cdot(\lambda_2, n_2)) = ([\lambda_1 + \lambda_2], 1)\cdot\pi((0, n_1)\cdot(0, n_2))$$

$$\pi((\lambda_1, n_1) - (\lambda_2, n_2)) = (\lambda_1, 1)\cdot\pi((0, n_1) - (\lambda_2 - \lambda_1, n_2)).$$

The table of moiular addition on mod(3+2i).

+	00	01	02	03	11	12	13	21	22	23	31	32	33
00	00	01	02	03	11	12	13	21	22	23	31	32	33
01	01	02	32	23	03	22	11	00	21	31	33	13	12
02	02	32	13	31	23	21	03	01	00	33	12	11	22
03	03	23	31	21	22	32	12	11	13	00	01	33	02

Unabbraviated varsion of the tables of overflow on mod(3+2i).

	01	02	03	11	12	13	21	22	23	31	32	33
01	0	0	0	0	0	0	0	0	0	0	0	0
02	0	33	01	0	03	11	0	13	21	0	23	31
03	0	01	0	0	11	0	0	21	0	0	31	0
11	0	0	0	0	0	0	0	0	0	0	0	0
12	0	03	11	0	13	21	0	23	31	0	33	01
13	0	11	0	0	21	0	0	31	0	0	01	0
21	0	0	0	0	0	0	0	0	0	0	0	0
22	0	13	21	0	23	31	0	33	01	0	03	11
23	0	21	0	0	31	0	0	01	0	0	11	0
31	0	0	0	0	0	0	0	0	0	0	0	0
32	0	23	31	0	33	01	0	03	11	0	13	21
33	0	31	0	0	01	0	0	11	0	0	21	0

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Abbreviatei/reduced version of the tables of overflow on mod (3+2i)

	01	02	03
01	0	0	0
02	0	33	01
03	0	01	0

Examples. a) to find product $< (2.\ 3) \times (3.\ 2) |_{p}^{-}$. According to formula (5.1), we have

$$< (2.\ 3)\cdot(3.\ 2) |_{p}^{-} = (| 2 + 3 |_{4}^{+} \cdot 1)\cdot < (0.\ 3) \times (0.\ 2) |_{p}^{-} =$$

$$= (1.\ 1)\cdot < (0.\ 3) \times (0.\ 2) |_{p}^{-}.$$

From the abbreviated/reduced table of modular multiplication we find that $< (0.\ 3) \times (0.\ 2) |_{p}^{-} = 2.1,$ therefore

$$< (2.\ 3) \times (3.2) |_{p}^{-} = (1.\ 1) \times (2.\ 1) = 3.\ 1.$$

b) To find sum $< (3.\ 1) + (3.\ 3) |_{p}^{-}$. According to formula (5.2) we will obtain

$$< (3.\ 1) + (2.\ 3) |_{p}^{-} = (3.\ 1)\cdot < (0.\ 1) + | 2 - 3 |_{4}^{+} \cdot 3 |_{p}^{-} =$$

$$= (3.\ 1) < (0.\ 1) + (3.\ 3) |_{p}^{-}.$$

From the abbreviated/reduced table of modular addition we have

$$< 0.\ 1 + 3.\ 3 |_{p}^{-} = 1.\ 2,$$

consequently,

$$< (3.\ 1) - (2.\ 3) |_{p}^{-} = (3.\ 1)\cdot(1.\ 2) = 0.2.$$

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Chapter 4.

POSITIONAL NUMERATION SYSTEMS WITH THE COMPOSITE BASES.

§1. General/common/total formulation of the problem.

Let be is preset sts.k.ch. z. Let us consider the set of all ts.k.ch. of the form

$$\xi_0 + \xi_1 p + \xi_2 p^2 + \ldots + \xi_{n-1} p^{n-1}. \qquad (1.1)$$

where each variable/alternating ξ_n passes value by certain p.s.v. on mod p.

Generally speaking mentioned p.s.v. can be distinguished between themselves, i.e., depend on index k(0⩽k⩽n-1). We will indicate that the set ts.k.ch. (1.1) forms range $D(p^n)$ ts.k.ch. on mod p^n, represented by positional code (1.1).

It is clear that the geometric configuration of range depends

both on the selection of basis/base p and on selection p.s.v. on mod p on each k-th digit.

The study of the positional numeration systems of a similar form understands the study first of all of the following general/common/total problems:

a) the ranges of the representation of numbers;

b) the determination of overflows with the addition;

c) the operation of addition and multiplication;

d) the operation of translation/conversion.

Specifically, from the point of view of these questions will be examined below some positional numeration systems.

§2. Composite version of binary number system.

From the fact that $\|\pm 1 \pm i\| = 2$, follows: the full/total/complete system of deductions on any of the bases/bases +-1+-i double-discrete. In other words, any of the sets

$$\{0,\ 1\},\ \{0,\ i\},\ \{0,\ -1\},\ \{0,\ -i\} \qquad (2.1)$$

can be chosen as p.s.v. on any of the moduli/modules +-1+-i.

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Because of double-discreteness p.s.v. on moduli/modules p=+-1+-i appears the possibility of the introduction of a certain analog of binary positional arithmetic in the ring of ts.k.ch.

Since numbers +-1+-i are associated, from the point of view of division theory construction of the positional numeration system with the bases/bases +-1+-i is represented by adequate procedure. However, the algorithms of arithmetic operations can be changed in the dependence on that, which of sets (2.1) will be chosen as the basis for the image bit digits with one of the bases/bases of form p=+-1+-i. The latter is connected with the fact that the character of additive and multiplicative overflows can depend on the selection of p.s.v.

Thus, from the positions of the analysis of multiplicative overflow is profitable to select as p.s.v. set { 0, 1 }, since in the class of these deductions are absent multiplicative overflows. In connection with this it is expedient to consider all possible versions of the binary coding of ts.k.ch.

Algorithms of the formation of the deductions of arbitrary ts.k.ch. on modulus/module $p=((-1)^k-(-1)^m i)$

Case a. Let as p.s.v. on mod $((-1)^k+(-1)^m i)$ be are chosen one of the sets { 0.1 }, { 0, -1 }. It is necessary for the the arbitrary ts.k.ch. a+bi to construct u, v ρ such, that

$$a+bi=(u-iv)\,p+\rho, \qquad (2.2)$$

where $\rho\in\{0,1\}$ $(\rho\in\{0,-1\})$.

From (2.2) it follows

$$\begin{cases}(-1)^k u-(-1)^m v=a-\rho,\\ (-1)^m u+(-1)^k v=b.\end{cases}$$

Hence

$$\begin{cases}u=\dfrac{(-1)^k a-(-1)^m b}{2}-(-1)^k\dfrac{\rho}{2},\\[2mm] v=\dfrac{(-1)^k b-(-1)^m a}{2}+(-1)^m\dfrac{\rho}{2}.\end{cases} \qquad (2.3)$$

Since numbers a, b and u, C - wholes, then from relationships/ratios (2.3) it follows

$$|\rho|=\begin{cases}0,\ \text{если}^{(1)}\ a\equiv b\ (\mathrm{mod}\ 2),\\ 1,\ \text{если}^{(1)}\ a\not\equiv b\ (\mathrm{mod}\ 2).\end{cases} \qquad (2.4)$$

Key: (1). if.

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Relationships/ratios (2.4) and (2.3) uniquely determine expansion

(2.2).

Case b. As p.s.v. on mod $((-1)^k + (-1)^m i)$ is selected one of the sets { 0, i }, { 0, -i }.

Then it is analogous with previous, arbitrary ts.k.ch. unambiguously is expanded according to the formula

$$a + bi = (u + iv)\, p + i\rho, \qquad (2.5)$$

where $\rho \in \{0, 1\}$ ($\rho \in \{0, -1\}$), moreover

$$\begin{cases} u = \dfrac{(-1)^k a + (-1)^m b}{2} - (-1)^m \dfrac{\rho}{2}, \\ v = \dfrac{(-1)^k b - (-1)^m a}{2} + (-1)^k \dfrac{\rho}{2} \end{cases} \qquad (2.6)$$

and

$$|\rho| = \begin{cases} 0, \text{ если}^{(1)}\ a \equiv b \pmod 2, \\ 1, \text{ если}\ a \not\equiv b \pmod 2. \end{cases} \qquad (2.7)$$

Key: (1). if.

Let us introduce the designations

$$\begin{aligned} u_{k,m} &= \left[\frac{(-1)^k a + (-1)^m b}{2}\right], \\ v_{k,m} &= \left[\frac{(-1)^k b - (-1)^m a}{2}\right], \end{aligned} \qquad (2.8)$$

then (2.3) and (2.6) it is possible to rewrite respectively in the following form:

$$\begin{cases} u = u_{k,m} + \frac{|\rho|}{2}(1-(-1)^k \operatorname{sign}\rho), \\ v = v_{k,m} + \frac{|\rho|}{2}(1+(-1)^m \operatorname{sign}\rho), \end{cases} \quad (2.3')$$

$$\begin{cases} u = u_{k,m} + \frac{|\rho|}{2}(1-(-1)^m \operatorname{sign}\rho), \\ v = v_{k,m} + \frac{|\rho|}{2}(1-(-1)^k \operatorname{sign}\rho). \end{cases} \quad (2.6')$$

Let us reduce to the table of the rule of formation u, v, ρ expansions of type (2.2) and (2.5) depending on the selection of basis/base $p=(-1)^k+(-1)^m i$ and set of p.s.v.

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p	u	v	p.s.v.
$+1+i$	u_{00}	$v_{00}+\rho$	$\{0,1\}$
	$u_{00}+\rho$	v_{00}	$\{0,-1\}$
	u_{00}	v_{00}	$\{0,i\}$
	$u_{00}+\rho$	$v_{00}+\rho$	$\{0,-i\}$
$-1+i$	$u_{10}+\rho$	$v_{10}+\rho$	$\{0,1\}$
	u_{10}	v_{10}	$\{0,-1\}$
	u_{10}	$v_{10}+\rho$	$\{0,i\}$
	$u_{10}+\rho$	v_{10}	$\{0,-i\}$
$-1-i$	$u_{11}+\rho$	v_{11}	$\{0,1\}$
	u_{11}	$v_{11}+\rho$	$\{0,-1\}$
	$u_{11}+\rho$	$v_{11}+\rho$	$\{0,i\}$
	u_{11}	v_{11}	$\{0,-i\}$
$1-i$	u_{01}	v_{01}	$\{0,1\}$
	$u_{01}+\rho$	$v_{01}+\rho$	$\{0,-1\}$
	$u_{01}+\rho$	v_{01}	$\{0,i\}$
	u_{01}	$v_{01}+\rho$	$\{0,-i\}$

Positional coding ts.k.ch. with basis/base $p=(-1)^k+(-1)^m i$. Algorithms (2.2) and (2.5) the disintegration of arbitrary ts.k.ch. make it possible to determine the positional binary of odes on any of the bases/bases $p=(-1)^k+(-1)^m i$.

The algorithm of the consecutive determination of digits $\varepsilon_0, \varepsilon_1, \varepsilon_2, \ldots$ binary representation ts.k.ch. a+bi

$$a+bi=\varepsilon_0+\varepsilon_1 p+\varepsilon_2 p^2+\varepsilon_3 p^3+\ldots+\varepsilon_k p^k+\ldots \quad (2.9)$$

is assigned by the following flow chart of the calculation (based on the example when as p.s.v. it is select-d set { 0, 1 } or { 0, 1 }).

$a_0=a$	$b_0=b$	$\lvert\varepsilon_0\rvert=\lvert a+b\rvert_2$
$a_1=\frac{(-1)^k a_0+(-1)^m b_0}{2}-$ $-(-1)^k\frac{\varepsilon_0}{2}$	$b_1=\frac{(-1)^k b_0-(-1)^m a_0}{2}+$ $+(-1)^m\frac{\varepsilon_0}{2}$	[illegible]
$a_2=\frac{(-1)^k a_1+(-1)^m b_1}{2}-$ $-(-1)^k\frac{\varepsilon_1}{2}$	$b_2=\frac{(-1)^k b-(-1)^m a_1}{2}+$ $+(-1)^m\frac{\varepsilon}{2}$	$\lvert\varepsilon_2\rvert=\lvert a_2+b_2\rvert_2$
. . .	. . .	. . .
$a_s=\frac{(-1)^k a_{s-1}+(-1)^m b_{s-1}}{2}-$ $-(-1)^k\frac{\varepsilon_{s-1}}{2}$	$b_s-\frac{(-1)^k b_{s-1}}{2}-$ $-\frac{(-1)^m a_{s-1}}{2}+(-1)^m\frac{\varepsilon_{s-1}}{2}$	$\lvert\varepsilon_s\rvert=\lvert a_{s-1}+b_{s-1}\rvert_2$
. . .	. . .	. . .

In this diagram of digit $\varepsilon_0, \varepsilon_1, \varepsilon_2, \ldots$ is determined positional code (2.9) ts.k.ch. a+bi, beginning from the low-order digit.

Observation. Here and subsequently it is assumed that the numbering of the bits of the binary code is led from the low-order digits to senior. Expression $|\varepsilon_s|$ represents absolute part of deduction ε_s.

Let us refine, at what step/pitch one should complete the described process of binary coding.

Let us introduce the designation: $q_s = a_s + b_s i$.

By construction

$$q_{s-1} = q_s p + \varepsilon_{s-1} \quad (s \geq 1).$$

If $\varepsilon_{s-1} = 0$, then

$$|q_{s-1}| = \sqrt{2}\,|q_s|$$

and, therefore,

$$|q_s| < |q_{s-1}|. \qquad (2.10)$$

But if $|\varepsilon_{s-1}| = 1$, then we will distinguish two cases depending on that, which of the following pairs of p.s.v. is accepted as the image of the digits of the binary coding:

1. { 0, 1 }, { 0, -1 };

2. { 0, i }, { 0, -1 }.

In the case of 1 we will have:

$$(a_{s-1}-\varepsilon_{s-1})+b_{s-1}i=q_s p,$$

whence

$$(a_{s-1}-\varepsilon_{s-1})^2+b_{s-1}^2=2(a_s^2-b_s^2).$$

Let us present the left side of the latter/last equality in the form

$$(a_{s-1}-\varepsilon_{s-1})^2+b_{s-1}^2=2(a_{s-1}^2-b_{s-1}^2)-$$
$$-((a_{s-1}+\varepsilon_{s-1})^2+b_{s-1}^2-2).$$

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Since a_{s-1}, b_{s-1} - integers and $|a_{s-1}+b_{s-1}|_2=1$, which follows from the fact that $|\varepsilon_{s-1}|=1$, the minimum value which can take the form

$$(a_{s-1}+\varepsilon_{s-1})^2+b_{s-1}^2 \qquad (2.11)$$

equally to 0. This value it reaches at the single points

$$a_{s-1}=-\varepsilon_{s-1},\ b_{s-1}=0.$$

The following in the value value which can take this form, is equal to 2. This value it reaches at points $a_{s-1}=0,\ b_{s-1}=\pm 1$.

Thus, for all points, different from those mentioned above, which let us name critical, the form

$$(a_{s-1}+\varepsilon_{s-1})^2+b_{s-1}^2>2$$

and therefore for these values a_{s-1}, b_{s-1}

$$|q_s| < |q_{s-1}|.$$

In the case of 2 we will have:

$$a_{s-1} + (b_{s-1} - \varepsilon_{s-1})i = q_s p,$$

hence

$$a_{s-1}^2 + (b_{s-1} - \varepsilon_{s-1})^2 = 2(a_s^2 + b_s^2)$$

or

$$a_s^2 + b_s^2 = a_{s-1}^2 + b_{s-1}^2 - \frac{a_{s-1}^2 + (b_{s-1} + \varepsilon_{s-1})^2 - 2}{2}.$$

Is analogous with that presented the minimum value, equal to 0, the form

$$a_{s-1}^2 + (b_{s-1} + \varepsilon_{s-1})^2$$

acquires at unique critical point $a_{s-1} = 0$, $b_{s-1} = -\varepsilon_{s-1}$, also value equal to 1, it takes at points $a_{s-1} = \pm 1$, $b_{s-1} = 0$.

Consequently, for all points, different from the critical ones, it is correct

$$|q_s| < |q_{s-1}|. \qquad (2.12)$$

Let us pause at the analysis of critical points.

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Them answer critical numbers -1, i, -i in the case of 1 and -i, +1, -1 in the case of 2. Let us note that

Binary code of number a+bi with basis/base p whose digits are represented as values 0.1;

the binary code of a number - a-bi with basis/base p whose digits are represented as values 0, -1;

the binary code of a number -b+ai with basis/base p whose digits are represented as values 0, i;

the binary code of number b-ai by basis/base p whose digits are represented as values 0, -i, they coincide.

In connection with this for each of the bases/bases $p=(-1^k+(-1)^m i$ it suffices to consider the system of deductions { 0, 1 }.

Let us introduce the following designation for the bases/bases in question:

$$p_\alpha = i^\alpha (1+i) \quad (\alpha = 0, 1\ 2, 3).$$

Minimum value quadratic form (2.11) accepts at points $a_{s-1}=-1,\ b_{s-1}=0.$ Therefore let us consider the translation/conversion of number -1 into the binary code.

Is obvious, $-1=-p_\alpha \cdot p_\alpha+1,$ but $-\bar{p}_\alpha=(-1)^\alpha i p_\alpha.$

Consequently,

$$-1 = i^{2\alpha+1} p_\alpha^2 + 1. \qquad (2.13)$$

By the following step/pitch in the dependence on the parity or the oddness α it is necessary to decompose into the binary code of a number +i and -i.

Let α - even (α=0.2). Then with α=0 we have

$$i = ip_0 + 1,$$

i.e. number i on mod (1+i) is modular. In other words, it can be represented by the positional code on basis p_0=1+i of the arbitrary length:

$$i = 1 + 1\cdot p_0 + 1\cdot p_0^2 + 1\cdot p_0^3 + \ldots + 1\cdot p_0^n + i\cdot p_0^{n+1}.$$

With a=2 we have

$$i = (-i)\, p_2 + 1, \; - i = 1\cdot p_2 + 1$$

or

$$i = 1\cdot p_2^2 + 1\cdot p_2 + 1.$$

Hence, by force (2.13), we consist that on basis/base p_2=-1-i number -1 is represented by the binary code of the finite length:

$$-1 = 1\cdot p_2^4 + 1\cdot p_2^3 + 1\cdot p_2^2 + 1\cdot p_2^0.$$

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Let α - odd (α=1, 3). Then with α=1 we have:

$$-i = i\cdot p_1 + 1, \;\; i = 1\cdot p_1 + 1$$

or

$$-i = 1\cdot p_1^2 + 1\cdot p_1 - 1.$$

Hence, by force (2.13), we consist that on basis/base $p_1=-1+i$ number -1 is also represented by the binary code of the finite length:

$$-1 = 1\cdot p_1^4 + 1\cdot p_1^3 + 1\cdot p_1^2 + 1.$$

With $\alpha=3$ we have

$$-i = (-i)\,p_3 + 1,$$

i.e. a number -1i on mod (1-i) is modular. Thus,

for bases/bases $p_1=-1+i$ and $p_2=-1-i$ critical numbers have a disintegration into the binary codes of finite length, and therefore on the strength of the fact that for all other numbers condition (2.10), any ts.k.ch. are carried out decomposed/expanded by unique form into the binary code of finite length on bases/bases $p_1=-1+i$, $p_2=-1-i$;

for bases/bases $p_0=1+i$, $p_3=1-i$ critical numbers have a disintegration into the binary code of arbitrary length. From evaluations (2.12) it follows that any ts.k.ch. is decomposed/expanded by unique form into the binary code on bases/bases $p_0=1+i$, $p_3=1-i$, the code having finite length, if the

process of consecutive indexing, described by diagram, is completed not by a critical number. The code can have arbitrary length, if in the process consecutively/serially of division on the diagram is encountered a critical number.

For the survey/coverage is given the table of the binary expansions of some constants in numeration system with the bases/bases of form $p_\tau = i^\tau(1-i)$ and with the selection as p.s.v. of one of sets (2.1). Moreover in all cases of binary coding nonzero digits are designated by ones, although by their nature they can be not unity. Here is used below the ordinary principle of the binary notation: the code is read from left to right; the first, different from zero, digit senior.

Table is given completely for all possible versions of binary coding, thanks to which more graphically are exhibited the invariants of the binary coding of ts.k.ch.

The analysis of table is shown:

1) the character of additive overflow does not depend on the method of assignment p.s.v.;

2) numeration system with bases/bases p_0=1+i, p_3=1-i possesses

modular numbers, the value of additive overflow having in the binary recording potentially infinite extent, which complicates the operation of addition;

3) the best version of binary number system is system with basis/base $p_1=-1+i$ (or $p_2=-1-i$) with the selection of p.s.v. $\{0, 1\}$ for the digital image of the bits of the binary code.

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Table of the binary disintegration of constants in terms of bases/bases p.

п. с. в. (1) / p_ν	{0,1}	{0,i}	{0,−1}	{0,−i}
$p_0=1+i$	$-1=(i)01$ $i=(i)1$ $-i=(i)011$ $-1+i=(i)10$ $-1-i=(i)010$ $1-i=(i)0110$ $2=(i)01100$	$-1=(-1)1$ $1=(-1)011$ $-i=(-1)01$ $-1+i=10$ $-1-i=(-1)10$ $1-i=(-1)010$ $1+i=(-1)0110$ $2i=(-1)01100$	$1=(-i)01$ $i=(-i)011$ $-i=(-i)1$ $-1+i=(-i)0101$ $-1-i=10$ $1-i=(-i)01$ $1+i=(-i)010$ $-2=(-i)01100$	$-1=(1)011$ $i=(1)01$ $1=(1)1$ $-1+i=(1)010$ $-1-i=(1)0110$ $1-i=10$ $1+i=(1)10$ $-2i=(1)01100$
$p_1=-1+i$	$-1=11101$ $i=11$ $-i=111$ $1+i=1110$ $-1-i=110$ $1-i=111010$ $2=1100$	$-1=11$ $1=111$ $-i=11101$ $-1+i=1110$ $-1-i=10$ $1-i=110$ $1+i=111010$ $2i=1100$	$1=11101$ $i=111$ $-i=11$ $-1+i=111010$ $-1-i=1110$ $1-i=10$ $1+i=110$ $-2=1100$	$-1=111$ $i=11101$ $1=11$ $-1+i=110$ $-1-i=111010$ $1-i\equiv1110$ $1+i=10$ $-2i=1100$
$p_2=-1-i$	$-1=11101$ $i=111$ $-i=11$ $1+i=111010$ $-1+i=110$ $1-i=1110$ $2=1100$	$-1=111$ $1=11$ $-i=11101$ $-1+i=111010$ $-1-i=110$ $1-i=10$ $1+i=1110$ $2i=1100$	$1=11101$ $i=11$ $-i=111$ $-1+i=1110$ $-1-i=111010$ $1-i=110$ $1+i=10$ $-2i=1100$	$-1=11$ $i=11101$ $1=111$ $-1+i=10$ $-1-i=1110$ $1-i=111010$ $1+i=110$ $-2i=1100$
$p_3=1-i$	$-1=(-i)01$ $i=(-i)011$ $-i=(-i)1$ $-1+i=(-i)010$ $-1-i=(-i)10$ $1+i=(-i)0110$ $2=(-i)01100$	$-1=(1)011$ $1=(1)1$ $-i=(1)01$ $-1+i=(1)0110$ $-1-i=(1)010$ $1-i=(1)10$ $1+i=10$ $2i=(1)01100$	$1=(i)101$ $i=(i)1$ $-i=(i)1011$ $-1+i=10$ $-1-i=(i)10110$ $1-i=(i)1010$ $1+i=(i)10$ $-2=(i)101100$	$-1=(-1)1$ $i=(-1)01$ $1=(-1)011$ $-1+i=(-1)10$ $-1-i=10$ $1-i=(-1)0110$ $1+i=(-1)010$ $-2i=(-1)01100$

Key: (1). p.s.v.

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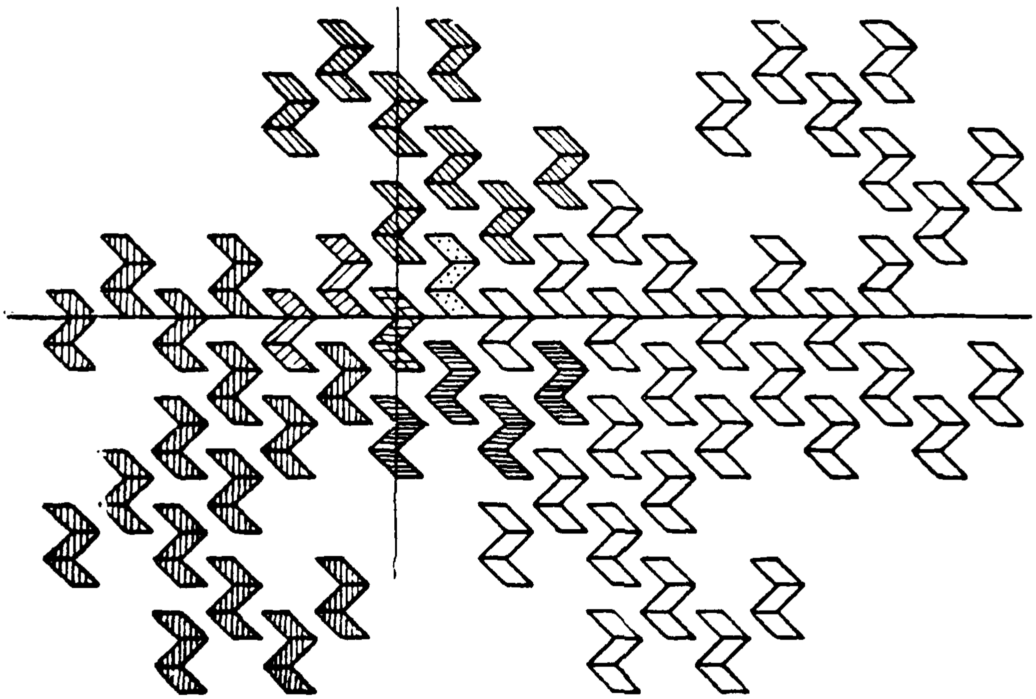

Fig. 29. Deductions on mod $(-1+i)^8$.

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§3. Positional arithmetic of numeration system with basis/base $p=-1+i$.

Above it was shown that on basis/base $p=-1+i$ any ts.k.ch. in a unique manner is decomposed/expanded in by binary the code of finite

length (Fig. 29). The translation algorithm of arbitrary ts.k.ch. a+bi the binary code is described by the following diagram consecutively/serially of division:

$$a_0 = a \qquad b_0 = b \qquad \varepsilon_0 = |a_0 - b_0|_2$$

$$a_1 = \frac{b_0 - a_0 + \varepsilon_0}{2} \quad b_1 = -\frac{b_0 + a_0 - \varepsilon_0}{2} \quad \varepsilon_1 = |a_1 + b_1|_2$$

$$a_2 = \frac{b_1 - a_1 + \varepsilon_1}{2} \quad b_2 = -\frac{b_1 + a_1 - \varepsilon_1}{2} \quad \varepsilon_2 = |a_2 + b_2|_2$$

$$\dots\dots\dots \quad \dots\dots\dots \quad \dots\dots\dots$$

$$a_k = \frac{b_{k-1} - a_{k-1} + \varepsilon_{k-1}}{2} \quad b_k = -\frac{b_{k-1} + a_{k-1} - \varepsilon_{k-1}}{2} \quad \varepsilon_k = |a_k + b_k|_2$$

$$\dots\dots\dots \quad \dots\dots\dots \quad \dots\dots\dots$$

$$a + bi = \varepsilon_0 + \varepsilon_1 p + \varepsilon_2 p^2 + \dots + \varepsilon_k p^k + \dots$$

Example. To represent ts.k.ch. 3+4i in the binary code on basis/base p=-1+:

$$a_0 = 3 \qquad b_0 = 4 \qquad \varepsilon_0 = |a_0 + b_0|_2 = 1$$

$$a_1 = \frac{4-3+1}{2} = 1 \qquad b_1 = -\frac{4+3-1}{2} = -3 \qquad \varepsilon_1 = 0$$

$$a_2 = \frac{-3-1+0}{2} = -2 \qquad b_2 = -\frac{-3+1-0}{2} = 1 \qquad \varepsilon_2 = 1$$

$$a_3 = \frac{1-(-2)+1}{2} = 2 \qquad b_3 = -\frac{1+(-2)-1}{2} = 1 \qquad \varepsilon_3 = 1$$

$$a_4 = \frac{1-2+1}{2} = 0 \qquad b_4 = -\frac{1+2-1}{2} = -1 \qquad \varepsilon_4 = 1$$

$$a_5 = \frac{-1-0+1}{2} = 0 \qquad b_5 = -\frac{-1+0-1}{2} = 1 \qquad \varepsilon_5 = 1$$

$$a_6 = \frac{1-0+1}{2} = 1 \qquad b_6 = -\frac{1+0-1}{2} = 0 \qquad \varepsilon_6 = 1$$

$$a_7 = \frac{0-1+1}{2} = 0 \qquad b_7 = -\frac{0+1-1}{2} = 0$$

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Thus,

$$3+4i=\varepsilon_6\varepsilon_5\varepsilon_4\varepsilon_3\varepsilon_2\varepsilon_1\varepsilon_0=1111101.$$

For the restoration/reduction ts.k.ch. on the basis of its binary code, it is desirable to have a table of degrees of basis/base p=-1+i. In that case the unknown value can be obtained and the sum of those degrees p, for which discharging digit is different from 0.

Since $(-1+i)^2=-2i$; $(-1+i)^3=2(1+i)$; $(-1+i)^4=-2^2$, then

$$(-1)i)^n=(-1+i)^{\left[\frac{n}{4}\right]\cdot 4+|n|_4}=(-2^2)^{\left[\frac{n}{4}\right]}(-1+i)^{|n|_4};$$

i.e.

$$(-1+i)^n=(-2^2)^{\left[\frac{n}{4}\right]}\begin{cases}1, & |n|_4=0\\ -1+i, & |n|_4=1\\ -2i, & |n|_4=2\\ 2(1+i), & |n|_4=3\end{cases}$$

Example. To restore/reduce ts.k.ch. on the basis of its binary code 1 1 1 1 1 0 1.

(1) Имеем степень	0:	$1 \to 1$
»	1:	$0 \to 0$
»	2:	$1 \to -2i$
»	3:	$1 \to 2+2i$
»	4:	$1 \to -2^2$
»	5:	$1 \to 2^2-2^2i$
»	6:	$1 \to \frac{2^3i}{3+4i}$

Key: (1). We have a degree.

Operations of addition, subtraction, multiplication. The operations of addition and multiplication of the numbers, preset by the binary code, are formed/shaped in accordance with the operation of addition and multiplication of deductions.

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It is clear that the operation of the addition of the complex numbers, represented by the binary code, will possess distinctive specific character against the ordinary binary arithmetic of real numbers, since complex numbers, being by their nature two-dimensional vector quantities, are represented by the one-dimensional code.

So the correctly following sentence: the operation of the addition of the complex numbers, preset by the binary code, generally

speaking, it cannot be made via formal step-by-step addition taking into account transfers from the low-order digits into the senior.

Let us confirm the sentence indicated. Let us find the sum of numbers ip=110, i=11.1 (Fig. 30). After the first stroke/cycle of addition low-order digit taking into account transfers we will have:

$$\begin{array}{rl} 110\,\rceil & \\ 110 & = ip \\ 11 & = i \\ 1 & = 1 \\ \hline 0 & \end{array}$$

As a result of the first stroke/cycle of addition initial situation - the requirement to sum up the numbers

$$\begin{array}{r} 110 \\ 11 \\ 1 \end{array}$$

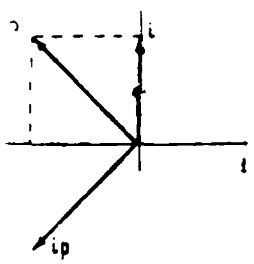
was not changed. It will not be changed also after the second stroke/cycle of addition.

the table of the addition of deductions.

+	0	1
0	0	1
1	1	1100

Multiplication table of deductions.

×	0	1
0	0	0
1	0	1



Key: (1). Fig. 30. ip+i+1=0.

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Consequently, the formalism of the operation of the transfer in this case will lead to the unlimited continuation of the process of addition.

Thus, for executing the operation of addition of ts.k.ch. preset by the binary code, besides the strokes/cycles of addition taking into account transfers from the low-order digits into the senior, is necessary certain pattern logic on the recognition of the cycle of components/terms/addends whose sum is equal to zero.

The elementary zero cycle of this kind (i.e. by the cycle, which contains the shortest binary words) is the cycle

$$\begin{array}{r} -i = 111 \\ +\ i = \ \ 11 \\ \hline 0\ \ 000 \end{array}$$

Its varieties they are

$$\begin{array}{r} 110 \\ 11 \\ 1 \\ \hline 000 \end{array} \quad \begin{array}{r} 101 \\ 10 \\ 11 \\ \hline 000 \end{array} \quad \begin{array}{r} 100 \\ 11 \\ 11 \\ \hline 000 \end{array} \quad \begin{array}{r} 100 \\ 11 \\ 10 \\ 1 \\ \hline 000 \end{array} \quad \begin{array}{r} 111 \\ 10 \\ 1 \\ \hline 000 \end{array}$$

The correctly following rule: let it be it summarized a ts.k.ch. represented by the binary code; any k of the pairs of unity s-th and (s+1)-th of bits and k units, (s+2)-th bit of components/terms/addends (if such combinations of unity there exist) is formed zero cycle.

Example 1. Minimum zero cycles are outlined.

```
    11 ←     (1) второй перенос
     11 ←    (2) первый перенос
1110100000   (−1) (−1+i)^4
  11101000   (−1) (−1+i)^3
   1110100   (−1) (−1+i)^2
     11101   (−1)
----------------------------
      0001       1
```

Key: (1). the second transfer. (2). first transfer.

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The same example, but in other version of the formation of the zero cycles:

example 2.

```
1110100000          11 ←
 111101000            1110111   = 3i
  1110100           111010111   = 4 − i
    11101          --------------------
----------            1100100   = 4+2i
        01
```

The operation of subtraction can be carried out by means of the addition in accordance with the table

−	0	1
0	0	1
1	$\bar{1}$	0

In contrast to the transfer 1 1 0 0, formulated in the case of

addition, in the case of subtraction is formed/shaped the transfer, equal to 1 1 1 0. Here $\bar{1}$=-1=1 1 1 0 1.

Example 1. example 2.

$$\begin{array}{r} 1100100 = 4+2i \\ -\ 1110111 = 3i \\ \hline 00\bar{1}00\bar{1}\bar{1} \\ 11101 \\ 11101 \\ 11101 \\ \hline 111010111 = 4-i \end{array}$$

$$\begin{array}{r} 1100100 = 4+2i \\ -\ 111010111 = 4-i \\ \hline \bar{1}\bar{1}01\bar{1}00\bar{1}\bar{1} \\ 11101 \\ 11101 \\ 11101 \\ 1 \\ 11101 \\ 11101 \\ \hline 1110111 = 3i \end{array}$$

Example 3. To construct the number, contrasted to a number 3i=1 1 1 0 1 1 1.

We have

$$\begin{array}{r} -3i = \bar{1}\bar{1}\bar{1}0\bar{1}\bar{1}\bar{1} \\ 11101 \\ 11101 \\ 11101 \\ 11101 \\ 11101 \\ 11101 \\ \hline 110011 = -3i \end{array}$$

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The operation of multiplication, the force of the absence of multiplicative overflow with the multiplication of different from

zero ones deductions 1•1=1 can reduced to the addition, similar this occurs in the case of the binary arithmetic of real numbers.

Example 1. To multiply numbers 1 0 1 1 0 1 1 1 and 1 0 1 1.

```
            10110111
                1011
      11
        11
         11
            10110111
           10110111
         10110111
      11111000101001
```

The operation of multiplication by i; initial binary number is multiplied 1 1.

Example 2. 1 0 0 1 1 0 1•i=1 1 1 0 1 1 0 1 1 1

```
     11
        11
        1001101
       1001101
     1110110111
```

The operation of multiplication on -i; initial binary number is multiplied by 1 1 1.

Example 3. 1 0 0 1 1 0 1•(-i)=1 1 1 1 1 0 0 1 1

```
     1001101
    1001101
   1001101
   111110011
```

§4. Binary coding of ts.k.ch. on the basis/base 1-i.

It is of interest, what effect on the binary number system of complex numbers proves to be the presence of modular numbers. Let us consider basis/base p=1-i. For basis/base p=1-i in the remainder class { . 0, 1 } a modular number is value (-i);

$$-i=(-i)(1-i)+1.$$

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The translation circuit of ts.k.ch. a+bi into the binary code on basis/base p=1-i (diagram of consecutive indexing into p) is assigned as follows:

$$\left.\begin{array}{lll} a_0=a & b_0=b & \varepsilon_0=|a_0+b_0|_2 \\ a_1=\left[\frac{a_0-b_0}{2}\right] & b_1=\left[\frac{a_0+b_0}{2}\right] & \varepsilon_1=|a_1+b_1|_2 \\ \dots\dots & \dots\dots & \dots\dots\dots \\ a_k=\left[\frac{a_{k-1}-b_{k-1}}{2}\right] & b_k=\left[\frac{a_{k-1}+b_{k-1}}{2}\right] & \varepsilon_k=|a_k+b_k|_2 \\ \dots\dots & \dots\dots & \dots\dots\dots \end{array}\right\} . \quad (4.1)$$

Hence

$$a+bi=\varepsilon_0+\varepsilon_1 p+\varepsilon_2 p^2+\dots+\varepsilon_{k-1} p^{k-1}+\dots.$$

inasmuch as critically numbers -1, i, -i are reduced to modular number (-i) during their disintegration into the binary of codes -1=(-i) 0 1, i=(-i) 0 1 1, -i=(-i) 1, then for the the arbitrary ts.k.ch. the process of consecutive indexing is completed in the n stage when partial quotient $a_N+b_N i$ proves to be equal either to +1 or -i.

Examples. To find binary notation with basis/base p=1-i of numbers 7+5i, 1-4i

a	b	ε
7	5	0
1	6	1
—3	3	0
—3	0	1
—2	—2	0
0	—2	0
1	—1	0
1	0	1

$7+5i=10001010$

a	b	ε
1	—4	1
2	—2	0
2	0	0
1	1	0
0	1	1
—1	0	1
—1	—1	0
0	—1	$(-i)$

$1-4i=(-i)\ 0110001$

A number 1-4i, being given to the binary of odes on basis/base p=1-i, in the high-order digit has digit (-i) - a modular number.

Thus, relative to the algorithm of division (4.1) the set of all ts.k.ch. is divided/marked off into two classes. First class includes all those numbers, for which consecutive indexing (translation/conversion into the binary code) on diagram (4.1) is completed by the partial quotient, equal to +1; the second class

includes all those numbers, for which consecutive indexing on diagram (4.1) is completed by the partial quotient, equal to (-i). Numbers of first class we will call positive, and the second class - negative.

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The intersection of the set of positive and negative numbers is empty, since in view of the uniqueness of the binary notation ts.k. determined by diagram (4.1), any ts.k.ch. (different from 0) can er positive or negative. It should be noted that entire the partial quotients of consecutive indexing on diagram (4.1), used to a positive number, are positive numbers, and the partial quotients of negative numbers are negative numbers.

Actually/really, Let ts.k.ch. a+bi - negative, then

$$a+bi=\sum_{k=0}^{N-1}\varepsilon_k(1-i)^k+(-i)\ (1-i)^N$$

and at m step/pitch (m<N) of consecutive indexing on diagram (4.1) the partial quotient be to take the form

$$\sum_{k=m+1}^{N-1}\varepsilon_k(1-i)^k+(-i)\ (1-i)^{N-m},$$

i.e. it will be a number negative.

As is known, the set ts.k.ch. of the form

$$\sum_{k=0}^{n-1} \varepsilon_k (1-i)^k,$$

where each ε_k can accept one of the values - 0 or 1, forms p.s.v. on mod $(1-i)^n$ (Fig. 31). With any n this p.s.v. will consist only from the positive numbers u, therefore, not with what n this system will contain, for example, such numbers as -1, i, -i. P.s.v. on mod $(1-i)^{n+1}$, containing both positive and negative numbers, are formed ts.k.ch. the form

$$\sum_{k=0}^{n-1} \varepsilon_k (1-i)^k + \Theta (1-i)^n, \qquad (4.2)$$

where each ε_k accepts one of the values - 0 or 1, and Θ - value of 0 or (-i).

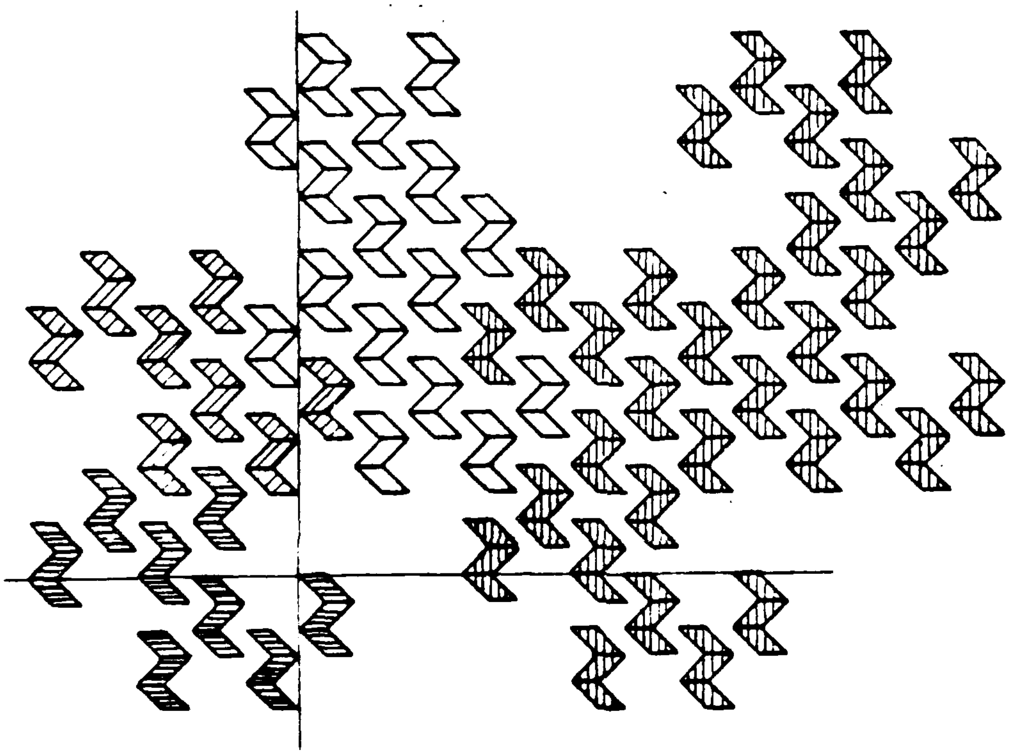

Fig. 31. P.s.v. on mod $(1-i)^8$.

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P.s.v. on mod $(1-i)^{n+1}$ form (4.2) we will call standardized and designate by symbol U_{n+1} (Fig. 32).

$z \in U_{n+1}$ and

$$z=\sum_{k=0}^{n-1} \varepsilon_k p^k+\Theta p^n.$$

Let us designate through z* inverted number, i.e., ts.k.ch. of form

$$z^*=\sum_{k=0}^{n-1} \varepsilon_k^* p^k+\Theta^* p^n,$$

where

$$\varepsilon_k^*=\begin{cases}1, \text{ если } \varepsilon_k=0 \\ 0, \text{ если } \varepsilon_k=1\end{cases}$$

$$\Theta^*=\begin{cases}0, \text{ если } \Theta=-i \\ -i, \text{ если } \Theta=0.\end{cases}$$

Key: (1) if.

Is obvious, $z^* \in U_{n+1}$, moreover z* - is negative, if z - is positive, and vice versa.

Let us find the sum of numbers z and z*:

$$z + z^* = \sum_{k=0}^{n-1} p^k + (-i)\, p^n = \frac{p^n - 1}{p-1} + (-i)\, p^n = i\,(p^n - 1) + $$
$$+ (-i)\, p^n = -i.$$

Thus,

$$z + z^* = -i. \qquad (4.3)$$

From relationship/ratio (4.3) it follows that z+i/2=-(z*+i/2), i.e., the integer points, which represent on the composite plane negative numbers, are symmetrical relative to the center of symmetry (0,-i/2) to the points, which represent positive numbers.

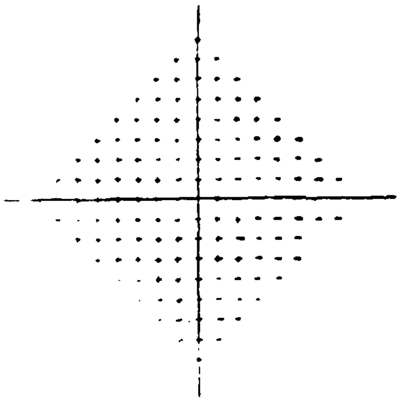

Fig. 32. Standardized p.s.v. cn mod $(1-i)^7$. (+) - positive numbers; (-) - negative numbers.

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From (4.3), it also follows that if is preset number z by its binary code, then in order to obtain the code of a number - z, it is necessary

a) to invert code z, i.e., to obtain z*; b) to the code of number z* to adjoin the code of number i.

Latter/last property is analogous with the property of the two's complement of negative numbers of the binary arithmetic of real numbers.

Thus, is observed the analogy of the properties of the binary coding of negative ones and positive ts.k.ch. and the properties of the binary coding of negative and positive ts.v.ch.

However, the arithmetic properties of positive and negative numbers do not apply to the class of positive and negative numbers.

Based on simple examples it is possible to ascertain that the sum (it is correct for the sum of positive numbers).

Example 1.

$$
\begin{array}{rrl}
 & & \overbrace{\hphantom{111\ldots1101}}^{n} \\
+ & -1 = (-i) & 111\ldots1101 \quad \text{(1) (негативное число)} \\
 & i = (-i) & 111\ldots1011 \quad \text{(негативное число)} \\
\hline
 & -1+i = (-i) & 111\ldots1010 \quad \text{(негативное число)}
\end{array}
$$

Key: (1) a negative number.

Example 2.

$$
\begin{array}{rrl}
 & & \overbrace{\hphantom{11\ldots10110}}^{n} \\
+ & 1+i = (-i) & 11\ldots10110 \quad \text{(1) (негативное число)} \\
 & -i = (-i) & 11\ldots11111 \quad \text{(негативное число)} \\
\hline
 & 1 = 0 & 00\ldots\quad 1 \quad \text{(2) (позитивное число)}
\end{array}
$$

Key: (1) a negative number. (2) a positive number.

The operation of addition in the binary number system with basis/base p=1-i is complicated by the fact that the value of transfer represents the negative number:

$$\begin{array}{l} \overbrace{1=000\ldots01}^{n} \text{ (позитивное число)} \\ +\\ 1=000\ldots01 \text{ (позитивное число)} \\ \hline 2=(-i)1\underbrace{\ldots11101100}_{n} \text{ (негативное число)} \end{array}$$

Key: (1) a positive number. (2) a negative number.

Similar to binary arithmetic with basis/base p=-1+i in the case in question for executing the operation of addition besides the formal addition of deductions taking into account transfers from the low-order digits into the senior are also necessary further procedures on the development/detection of zero cycles.

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Simplest of the zero cycles they are

$$\begin{array}{r} -1=(-i)\ 11\ldots1101 \\ +\quad 1=\ 0\ \ 00\ldots0001 \\ \hline 0=\ 0\ \ 00\ldots0000 \end{array} \qquad \begin{array}{r} i=(-i)\ 11\ldots1011 \\ +\ -i=(-i)\ 11\ldots1111 \\ \hline 0=0\ \ 00\ldots0000 \end{array}$$

$$\begin{array}{r} -1-i=-(i)\ 11\ldots1110 \\ +\quad i=(-i)\ 11\ldots1011 \\ 1=\ 0\ \ 00\ldots0001 \\ \hline 0=\ 0\ \ 00\ldots0000 \end{array}$$

The complicated character of the formation of transfers significantly destroys the possibility of the circuit realization of the algorithms of binary arithmetic with basis/base p=1-i.

§ 5. Positional numeration system with basis/base p=1+2i.

Modulus/module p=1+2i (Fig. 33) is remarkable fact that the different from zero elements/cells p.s.a.n.v. on mod (1+2i) are the dividers/denominators of unity in ring ts.k.ch.:

$$\langle \cdot \mid \bar{}_{1+2i} = \{0, 1, i, -1, -i\}$$

and, therefore, multiplicative overflow on modulus/module p=1+2i is always equal to zero, i.e., $\pi(\zeta_1 \cdot \zeta_2)=0$ for any deductions ζ_1, $\zeta_2 \in \langle \cdot \mid \bar{}_{1+2i}$.

Elementary operations on mod (1+2i) are implemented in accordance with the tables of modular addition, additive overflow and modular multiplication.

(1) Таблица модульного сложения

$+$	0	1	i	-1	$-i$
0	0	1	i	-1	$-i$
1	1	i	$-i$	0	-1
i	i	$-i$	-1	1	0
-1	-1	0	1	$-i$	i
$-i$	$-i$	-1	0	i	1

(2) Таблица аддитивного переполнения

$\pi(\xi_1+\xi_2)$	0	1	i	-1	$-i$
0	0	0	0	0	0
1	0	$-i$	1	0	$-i$
i	0	1	1	i	0
-1	0	0	i	i	-1
$-i$	0	$-i$	0	-1	-1

Key: (1). Table of modular addition. (2). Table of additive overflow.

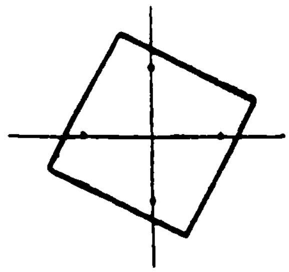

Fig. 33. P.s.a.n.v. on mod(1+2i).

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Modulus/module p=1+2i is the minimum odd modulus/module, which satisfies the criteria of the theorem of Gauss (3.2.5) about the isomorphism of composite deductions to real deductions. Diafantov equation, utilized for the search for the real deduction, to which is mapped the imaginary unit with the mentioned above isomorphism, takes the form

$$\lambda + 2\mu = 1.$$

Its particular solution is pair $\lambda=-1$, $\mu=1$. Therefore the unknown value ρ is equal to

$$\rho = |2\lambda - \mu|_5^+ = 2.$$

Hence the isomorphism between the composite deductions and the real deductions is assigned by the table

$\alpha+\beta i$	0	1	i	$-i$	-1
$\lvert\alpha+\rho\beta\rvert_5^+$	0	1	2	3	4

Таблица модульного умножения (1)

×	0	1	i	-1	$-i$
0	0	0	0	0	0
1	0	1	i	-1	$-i$
i	0	i	-1	$-i$	1
-1	0	-1	$-i$	1	i
$-i$	0	$-i$	1	i	-1

Key: (1). Table of modular multiplication.

Respectively the tables of addition and multiplication take the form:

(1) Таблица модульного сложения

+	0	1	2	3	4
0	00	01	02	03	04
1	01	32	13	31	00
2	02	13	14	00	21
3	03	34	00	41	42
4	04	00	21	42	23

(2) Таблица модульного умножения

×	0	1	2	3	4
0	0	0	0	0	0
1	0	1	2	3	4
2	0	2	4	1	3
3	0	3	1	4	2
4	0	4	3	2	1

Key: (1). Table of modular addition. (2). Table of modular multiplication.

Observation. The first digit of the element/cell of the table of addition indicates the value of overflow.

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For the representation of ts.k.ch. a+bi into the positional code with basis/base p=1=2i we will use, as usual, the algorithm of the consecutive division. For facilitating the calculation it is convenient to use the mentioned above isomorphism. Respectively algorithm accepts the form:

1) we determine the deduction of number a+bi on mod (1+2i). For this purpose we compute value

$$r_0 = | a_0 + b_0 \rho |_5 \quad (a_0 = a, \ b_0 = b)$$

and in value r_0 from the table we respectively restore composite deduction ζ_0;

2) we form the difference

$$(a_0 + b_0 i) - \zeta_0 = a_0' + b_0' i$$

and we compute partial quotient a_1+b_1i from division of a_0+b_0i into p:

$$a_1 + b_1 i = \frac{a_0' - b_0' i}{p} = \frac{a_0' - 2b_0'}{5} + i\frac{b_0 - 2a_0'}{5}.$$

Remainder/residue ζ_0 determines the low-order digit of the positional code with basis/base p=1+2i of number a+bi. For obtaining the following digit the described above computational process is repeated relative to number a_1+b_1i and so forth.

Example. Translate ts.k.ch. 6-16 i into the positional code with basis/base (1+2i). ρ=2.

a_k+b_ki	$\lvert a_k+\rho b_k\rvert_5$	ζ_k	a'_k	b'_k	$a'_k+2b'_k$	$b'_k-2a'_k$
$6-16i$	$\lvert 6-2\cdot 16\rvert_5=4$	-1	7	-16	-25	-30
$-5-6i$	$\lvert -5-2\cdot 6\rvert_5=3$	$-i$	-5	-5	-15	5
$-3+i$	$\lvert -3+2\cdot 1\rvert_5=4$	-1	-2	1	0	5
i	$\lvert 0+2\cdot 1\rvert_5=2$	i	0	0	0	0

Thus

$$6-16i = ip^3+(-1)p^2+(-i)p+(-1)$$

or

$$6-16i = 2p^3+4p^2+3p+4.$$

For the purpose of the opposite translation/conversion in certain cases it is convenient to use the diagram of Horner for calculating the value of polynomial at the preset point.

Let be preset ts.k.ch. by the positional code

$$\zeta_n p^n+\zeta_{n-1}p^{n-1}+\ldots+\zeta_1 p+\zeta_0.$$

We construct the sequence of the numbers

$$\zeta_n p+\zeta_{n-1}=q_{n-1}$$
$$\zeta_{n-1} p+\zeta_{n-2}=q_{n-2}$$
$$\cdots\cdots\cdots$$
$$\zeta_1 p+\zeta_0=q_0=a+bi.$$

The order of calculations along the diagram of Horner is usually placed as follows:

ζ_n	ζ_{n-1}	ζ_{n-2}	$\cdots$	ζ_2	ζ_1	ζ_0
$p=1+2i$	$\zeta_n p$	pq_{n-1}	$\cdots$	pq_3	pq_2	pq_1
	q_{n-1}	q_{n-2}		q_2	q_1	$q_0=a+bi$

Example. To restore/reduce ts.k.ch. on its positional code

$$ip^3+(-1)p^2+(-i)p+(-1):$$

i	-1	$-i$	-1
$1+2i$	$-2+i$	$-5-5i$	$7-16i$
	$-3+i$	$-5-6i$	$6-16i=a+bi$

The operations of addition and multiplication are implemented by traditional positional methods.

Example 1. To fulfill the operation of the service:

$$\begin{array}{r|ccccccl}
-9-19i= & & 1 & 0 & 1 & 1 & 3 & \\
16-11i= & + & 2 & 1 & 4 & 1 & 4 & \\
\hline
 & & 3 & 1 & 0 & 2 & 2 & \\
 & 1 & 0 & 0 & 3 & 4 & & \text{переносы от поразрядного суммирования (1)} \\
\hline
 & 1 & 3 & 1 & 3 & 1 & 2 & \\
 & 0 & 0 & 0 & 2 & & & \text{переносы (2)} \\
\hline
 & 1 & 3 & 1 & 0 & 1 & 2 & = +7-30i
\end{array}$$

Key: (1) transfers from step-by-step addition (2) transfers.

Example 2.

$$\begin{array}{r|ccccccl}
4+31i= & & 4 & 3 & 2 & 3 & 1 & \\
-3-33i= & + & 1 & 2 & 0 & 1 & 1 & \\
\hline
 & & 0 & 0 & 2 & 4 & 2 & \\
 & & & & 3 & 3 & & \text{переносы (1)} \\
\hline
 & & & & 0 & 2 & 2 & \\
 & & & & 4 & & & \text{переносы (1)} \\
\hline
 & & & & 4 & 2 & 2 & =1-2i
\end{array}$$

Key: (1) transfers.

Since on basis/base p=1+2i in the class a.n.v. are absent the multiplicative overflows, the operation of multiplication is reduced

to the stroke/cycle of step-by-step multiplication, the appropriate shift/shear of the obtained products with the subsequent addition.

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Example. To compute product ts.v.ch. - 1+4i and -4-i:

	$-1+4i = 1\,3\,2$	
	$-4-i = 2\,1\,4$	
	4 2 3	поразрядное умножение на 4 (1)
+	1 3 2	поразрядное умножение на 1 со сдвигом (2)
+	1 2 4 3	
	4 1	переносы (3)
+	0 3 4 3	
	1	переносы
	1 3 4 3	(2)
+	2 1 4	поразрядное умножение на 2 со сдвигом
+	2 2 2 4 3	
	3 4	переносы (3)
+	0 1 2 4	
	2	переносы
	$2\,1\,2\,4\,3 = 8-15i$	

Key: (1) step-by-step multiplication by 4. (2) step-by-step multiplication by 1 with the shift/shear. (3) transfers.

Observation. Since the real and alleged parts of basis/base p=1+2i are mutually simple, has the capability to shorten the multiplication tables and addition and thereby it is essential to facilitate the circuit realization of executing the operations. In this case, since different from zero deductions in mod(1+2i) are exact four, the special coding (introduced in chapter 3, § 5) of deductions by means of the rank and the mantissas in this case leads to the fact that all deductions will have one and the same mantissa, i.e., for the code of deduction it is possible to take its rank. Then

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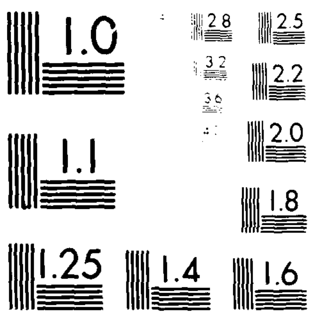

MICROCOPY RESOLUTION TEST CHART

multiplication table is reduced to summation over the modulus/module of four ranks of cofactors, and the addition of deductions with the overflows - to the table of the reccdings of deductions.

Actually/really, let us code deductions as follows:

α	0	1	i	-1	$-i$
(1) Код	$\bar{0}$	0	1	2	3

Key: (1). Code.

Let us compare to each deduction its rank.

Then the operation of the multiplication of deductions will be implemented in accordance with the rules:

1) $\bar{0}\cdot\alpha=\bar{0}$ for any deduction α; 2) if α_1, $\alpha_2\neq\bar{0}$, then $\alpha_1\cdot\alpha_2=|\alpha_1+\alpha_2|_4^+$.

The operation of addition can be reduced to the operation of multiplication and the recoding

$$\alpha_1 \ominus \alpha_2 + \alpha_1\cdot(0 \ominus |\alpha_2-\alpha_1|_4).$$

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Here operation $\oplus$ answers modular summation over mod (1+2i). Function $0\ominus+\beta=<0+\beta=|_p^- +\delta\cdot p$ is realized by recoding, according to

the table of its values

$\oplus$	$\bar{0}$	0	1	2	3
0	0	$\underline{3}1$	03	$\bar{0}$	$\underline{3}2$

In the table the first (emphasized) digit of a two-place number indicates the value of overflow.

§ 6. Positional numeration systems of complex numbers with bases/bases p=+2 and p=-2.

Numeration system with basis/base p=2. Since for any ts.k.ch. a+bi is correct

$$a+bi=\left(\left[\frac{a}{2}\right]+i\left[\frac{b}{2}\right]\right)2+(\,|a|_2+i\,|b|_2\,|_2). \quad (6.1)$$

then p.s.v. on mod 2 it is formed by the set of the numbers

$$<\cdot\,|_2^+=\{0,\ 1,\ i\ 1+i\}. \quad (6.2)$$

It is obvious, the translation/conversion of number a+bi into the positional code on mod 2 is equivalent to the independent translation/conversion into the binary code of the real and alleged parts:

$$a+bi=\sum_{k=0}^{n}\varepsilon_k 2^k+i\sum_{k=0}^{m}\omega_k 2^k \quad (6.3)$$

or

$$a+bi=\sum_{k=0}^{\max(n,\,m)}(\varepsilon_k+i\omega_k)2^k.$$

Thus, positional coding ts.k.ch. on basis/base 2

$$a+bi=\sum_{k=1}^{n}\zeta_k 2^k, \quad (6.4)$$

where

$$\xi_k \in \langle \cdot | _2$$

it is equivalent to the representation of real and alleged parts ts.k.ch. by the binary code.

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In both cases the range of the numbers, represented by code (6.3) and code (6.4), is determined by the inequality

$$0 < a < 2^n$$
$$0 < b < 2^n.$$

From the positions of machine arithmetic (6.3) and (6.4) cannot be identified. Nevertheless, is used below positional representation of ts.k.ch. in the form (6.4), where the composite digits are coded as follows:

$$\begin{array}{c|cccc} a+bi & 0 & 1 & i & 1+i \\ \hline z & 00 & 01 & 10 & 11 \end{array}. \qquad (6.5)$$

This approach makes it possible, at least, to obtain the new algorithm of multiplication ts.k.ch. against the traditional multiplication of real and alleged parts with their subsequent addition and subtraction.

Sign form of the representation of ts.k.ch. In the principle are possible two versions of sign representation ts.k.ch.:

1) a+bi,

where

$$-2^n \leqslant a < 2^n,$$
$$-2^n \leqslant b < 2^n; \qquad (6.6)$$

where 2) $a+bi = i^\alpha (a' + b' i)$,

$$0 \leqslant \alpha \leqslant 3,$$
$$0 \leqslant a' < 2^n,$$
$$0 \leqslant b' < 2^n.$$

Let us pause at first at the first version. Let us show that the diagram of consecutive indexing into modulus/module 2 according to formula (4.1) of arbitrary ts.k.ch. from the square

$$-2^{n+1} < a < 2^{n+1}, \; -2^{n+1} \leqslant b < 2^{n+1}$$

leads to the analysis of the modified two's complement.

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This follows of two facts:

1) a number - 1 is modular with respect to p.s.v. (6.2) on mod 2, i.e.,

$$(-1) = (-1)\cdot 2 + 1;$$

2) for any a+bi $\in \langle\cdot|^{-}_{2^{n+2}}$ is correct

$$\langle\langle a+bi \,|^{-}_{2^{n+2}} |^{+}_{2^{n+2}} = \langle a+bi \,|^{+}_{2^n} +$$
$$+ \zeta_n \cdot 2^n + \zeta_{n+1} 2^{n+1}, \qquad (6.7)$$

where

$$\zeta_n, \zeta_{n+1} \in \langle\cdot|^{+}_{2}.$$

The representation of range $<a+bi|^{-}_{2^{n+2}}$ onto range $<a+bi|^{+}_{2^{n+2}}$ in accordance with formula (6.7) it is depicted on the diagrams (Fig. 34, 35).

Squares 0, 5, 10, 15 in figure 34 represent p.s.a.n.v. on mod 2^{n+1}, the others - zone of shaping of additive overflow. Figure 35 gives the forms of the squares, to which are mapped the squares p.s.v. $<a+bi|^{-}_{2^{n+2}}$ in accordance with formula (6.7).

Thus, taking into account coding (6.5) deductions p.s.v. (6.2), we consist that the algorithm of consecutive indexing into 2 mutually unambiguously compares arbitrary ts.k.ch. a+bi from the square

$$-2^{n+1}\leqslant a<2^{n},\quad -2^{n+1}\leqslant b<2^{n+1}$$

the binary coda with a length of $2^{2(n+2)}$, which represents certain point of square $<a+bi|^{+}_{2^{n+2}}$.

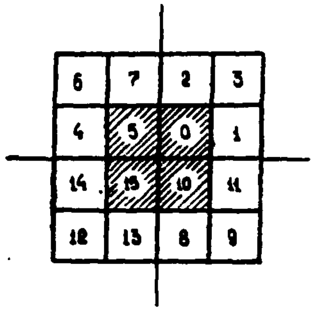

Fig. 34.

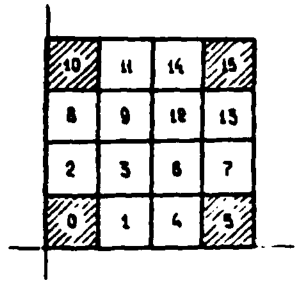

Fig. 35.

Fig. 34. Range <a+ bi | $\bar{}_{2}$n+2.

Fig. 35. Range <a+ bi | $\bar{}_{2}$n+2.

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Let us agree the numbering of the digits to produce toward the increase of the weights of digits; entire the binary word to divide/mark off on (n+1) the pairs of the bits, in each of which to distinguish digits with the even and odd number (low-order digit has a number 0). Than two pairs of the high-order digits are sign, that remained - digits of valency.

Let us give the table of sign situations. Let us preliminarily note that symbols sign (Re) and sign (Im) designate the sign of the real and alleged parts of the complex number, and ϕ(Re) and ϕ(Im) - overflow attributes on the real and alleged syllables, moreover if

ϕ=+1, then overflow to the positive side, and if ϕ=-1, then overflow to the negative side.

We see that the modified two's complement for ts.k.ch. can be obtained by the translaticn/conversion of real and alleged syllables independently into the modified two's binary complement, the modified code of alleged syllable cccupying digits with the odd numbers of the binary code of a complex number, and modified - digits with the even ones.

As is known, for the operation of addition in the binary arithmetic most convenient is the modified two's complement. However, it is inconvenient for executing the operation of the multiplication where the large convenience possesses the true representation. Analogous circumstance occurs alsc in the composite case.

Table of sign situations.

	Sign(Re)	Sign(Im)	φ(Re)	φ(Im)		Sign(Re)	Sign(Im)	φ(Re)	φ(Im)
0000	+	+	0	0	0110	−	+	−1	+1
0101	+	−	0	0	0111	−	+	0	+1
1010	−	+	0	0	1000	+	−	0	−1
1111	−	−	0	0	1001	+	−	+1	−1
0001	+	+	+1	0	1011	+	−	+1	0
0010	+	+	0	+1	1100	−	−	−1	−1
0011	+	+	+1	+1	1101	−	−	0	−1
0100	−	+	−1	0	1110	−	−	−1	0

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Let us consider the second version of sign representation of the number:

$$\text{if} \quad a+bi \in \langle \cdot \mid ^{-}_{1+2i}, \ \text{then}$$
$$a+bi = i^{\alpha}(a'+b'i),$$

where α=0, 1, 2, 3 and $0 \leq a'$, $b' < 2^{n}$. In that case, by analogy with binary arithmetic, we will indicate that the information about the sign (i.e. value α) and the binary code of number a'and b'i form in the set the true representation ts.k.ch. a+bi.

The conversion of the true representation into that modified is realized on the basis of the relationships/ratios

$$(a+bi)\,i = -b+ai,$$
$$(a+bi)\,i^2 = -a-bi,$$
$$(a+bi)\,i^3 = b-ai.$$

Let number a+bi be is preset by the true representation

$$a+bi=\overbrace{\varepsilon_{2n+1},\ \varepsilon_{2n}}^{n},\ \overbrace{\varepsilon_{2n-1},\ \varepsilon_{2n-2}}^{n-1},\ldots,\ \overbrace{\varepsilon_{3},\ \varepsilon_{2}}^{1},\ \overbrace{\varepsilon_{1},\ \varepsilon_{0}}^{0}.$$

Then

with $$\varepsilon_{2n+1}=0,\ \varepsilon_{2n}=0$$

the initial true representation coincides with that modified;

with $$\varepsilon_{2n+1}=0,\ \varepsilon_{2n}=1$$

the modified code takes the form

$$a+bi=0,\ 1,\ \varepsilon_{2n-2},\ \bar{\varepsilon}_{2n-1},\ \varepsilon_{2n-4},\ \bar{\varepsilon}_{2n-3},\ldots,\ \varepsilon_{2},\ \bar{\varepsilon}_{3},\ \varepsilon_{0},\ \bar{\varepsilon}_{1},$$

where the code

$$1,\ \bar{\varepsilon}_{2n-1}\ \bar{\varepsilon}_{2n-2},\ldots,\ \bar{\varepsilon}_{3},\ \bar{\varepsilon}_{1},$$

obtained by the ordinary modification of the binary code (real number),

$$1,\ \varepsilon_{2n-1},\ \varepsilon_{2n-3},\ldots,\ \varepsilon_{3},\ \varepsilon_{1};$$

with $$\varepsilon_{2n+1}=1,\ \varepsilon_{2n}=0$$

the modified code takes the form

$$a+bi=1,\ 1,\ \bar{\varepsilon}_{2n-1},\ \bar{\varepsilon}_{2n-2},\ldots,\ \bar{\varepsilon}_{3},\ \bar{\varepsilon}_{2},\ \bar{\varepsilon}_{1},\ \bar{\varepsilon}_{0},$$

where

$$1,\ \bar{\varepsilon}_{2n-1},\ \bar{\varepsilon}_{2n-3},\ldots\ \bar{\varepsilon}_1,$$
$$1,\ \bar{\varepsilon}_{2n-2},\ \bar{\varepsilon}_{2n-4},\ldots,\ \bar{\varepsilon}_0$$

the codes, obtained by the ordinary modification of the binary codes

$$1,\ \varepsilon_{2n-1},\ \varepsilon_{2n-3},\ldots,\ \varepsilon_1,$$
$$1,\ \varepsilon_{2n-2},\ \varepsilon_{2n-4},\ldots,\ \varepsilon_0;$$

with $\varepsilon_{2n+1}=1\ \ \varepsilon_{2n}=1$

the modified code takes the form

$$a+bi=1,\ 0,\ \bar{\varepsilon}_{2n},\ \varepsilon_{2n-1}\ \ \bar{\varepsilon}_{2n-2},\ \varepsilon_{2n-3},\ldots,\ \bar{\varepsilon}_0,\ \varepsilon_1,$$

where

$$1,\ \bar{\varepsilon}_{2n},\ \bar{\varepsilon}_{2n-2},\ \ldots,\ \bar{\varepsilon}_0$$

the code, obtained by the ordinary modification of the binary code

$$1,\ \varepsilon_{2n},\ \varepsilon_{2n-2},\ldots,\ \varepsilon_0.$$

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On the connection of the modified code with the reverse we here stop will not be.

Passing to the operation of addition, let us note that the coding of ts.k.ch. in the form (6.4) introduces nothing new in the operation addition against the fact that we have with the coding of

ts.k.ch. in the form (6.3).

Operations of multiplication. It is assumed that for the multiplication is used the true representation of ts.k.ch.

Multiplication table taking into account overflow takes the form

$\times$	0	1	i	$1+i$
0	0	0	0	0
1	0	1	i	$1+i$
i	0	i	-1	$(-1)2+(1+i)$
$1+i$	0	$1+i$	$(-1)2+(1+i)$	$i2+0$

From the table it follows that the multiplicative possible overflow of two types: i and of -1.

Since the binary code of number i takes form 10, overflow 1 appears in the odd bit of the binary code ts.k.ch. and the corresponding transfer extends similarly to ordinary binary addition, affecting odd bits. On the other hand, since number -1 - real and modular (i.e. its binary notation can be unlimitedly increased), negative overflow can arise only in the even bit of the binary code, and the corresponding transfer, extending, it affects only even bits, moreover the character of the effect of negative transfer is opposite to the character of the effect of positive transfer. Actually/really, let with the addition of two numbers, preset by the binary code, in the j bit arise the positive transfer, then 1) in the j bit is established/installed value of 0, 2) if in j+1 bit it was located 1,

then as a result of the addition of the transfer is established/installed value of 0, and the appearing transfer it affects the next decade according to the described principle;

if in j+1 bit it was located by 0, then as a result of the addition of the transfer is established/installed value 1 and further transfer does not extend.

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The effect of negative transfer is directly opposite: let in the j bit arise the negative transfer, then

1) in the j bit is established/installed value 1,

2) if in j+1- bit it was located 1, then as a result of the addition of the transfer is established/installed value of 0 and further transfer it does not extend;

but if in j+1 bit it was located by 0, then as a result of the addition of the transfer is established/installed value 1, but emergent negative transfer (-1) affects the next decade according to the described principle.

Before formulating the algorithm of multiplication as a whole, let us note the validity of the following sentence:

if ts.k.ch. a+bi (a⩾0, b⩾0), represented by the binary code, is multiplied by deduction $\zeta \in \langle \cdot | \,^{+}_{2}$, registered by the binary code, then the obtained as a result binary notation of product is the modified code of product.

The validity of assertion follows from the fact that as a result of the multiplication of positional code (6.4) by the deduction ζ we will obtain in accordance with the described above rule the positional code of form (6.4), in this case the emergent negative transfer can prove to be more significant digit.

Observation. The appearing negative transfer can prove to be only in odd sign position of a number. In even sign position will be always located digit 0, since positive transfer cannot change sign situation. The latter also follows from the fact that with a, b>0:

$$I_m((a+bi)^i_{1+i}) > 0.$$

From the fact that in even sign position of product always must be located digit 0, it follows that the length of the code of product $(a+bi)\zeta (\zeta \in \langle \cdot | \,^{+}_{2})$ without sign positions is determined by the sign/criterion in which the odd bit of senior pair must be equal to 1.

Multiplication table on mod 2 with the overflows.

×	00	01	10	11
00	00	00	00	00
01	00	001	10	11
10	00	10	0(−1)01	0(−1)11
11	00	11	0(−1)11	10 00

Examples.

$(7+5i)\times i=-5+7i=$
$=0(-1),\ 101111$

```
   11 01 11
×        10
   --------
   11 10 11       (1)
   ↑0↑1 —1   перенос }
 ←—1          перенос }
0(—1) 10 11 11     (2)
↑
└—знаковые разряды
```

$(7+5i)\ (1+i)=2+12i=$
$=10\ 10\ 01\ 00$

```
   11 01 11
×        11
   --------
   00 11 00
   ↑1 ↑0 ——перенос } (1)
 1—          перенос }
00, 10 10 01 00   (2)
↑
└—знаковые разряды
```

Key: (1) transfer. (2) sign positions.

On the basis of that presented it does not comprise the work to describe general/common/total multiplying circuit ts.k.ch., preset by the positional code on mod 2.

Example. To find product $(7+5i)\cdot(2+3i)=-1+31i$.

```
      ×  11 01 11
            11 10
        ----------------
    0(—1) 10 11  11
  10   10   01 00
  ----------------------
0(—1) 11   11   11 11 11
—1+3i=01  1111111111
```

Positional numeration system with basis/base p=-2. As the basis of the translation/conversion of ts.k.ch. into the positional code on

modulus/module p=-2 serves the formula

$$a+bi=\left(\left[\frac{a}{2}\right]+\left[\frac{b}{2}\right]i\right)(-2)+(|a|_2+i|b|_2) \quad (6.8)$$

and, therefore, the digits of the positional code are determined by the elements/cells of p.s.v.

$$\varepsilon<\cdot|^{+}_{-2}=\{0,\ 1,\ i,\ 1+i\}.$$

The deductions indicated we will code in accordance with Table (6.5).

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Example. Relying on formula (6.8) and using a diagram of consecutive indexing, to translate ts.k.ch. 3+7i into the positional code:

a	b	ε
3	7	11
−1	−3	11
+1	+2	01
0	−1	10
0	1	10
0	0	

$3+7i=10\,10\,01\,11\,11$

The advantage of numeration system with basis/base p=-2 in the comparison with the system with basis/base p=+2 lies in the fact that for basis/base p=-2 number -1 is not modular:

$$-1=1(-2)+1,$$

thanks to which, sign situations directly "inscribe" in the digits of the positional code of a number and do not require in connection with this additional procedures.

Deficiency/lack is the fact that range $<\cdot\,|_{-2^{n+1}}$ of numbers of the form

$$a+bi=\sum_{k=0}^{n} \zeta_k(-2)^k \quad \zeta_k \in <\cdot\,|\,\bar{}_{-2}$$

is not symmetrical to the origin cf coordinates (Fig. 36, 37), more exact it is determined by the system of the inequalities: with n the even

$$-\sum_{i=1}^{n/_2-1} 2^{2i-1} \leqslant a,\ b \leqslant \sum_{i=0}^{n/2} (-2)^{2i},$$

with n odd

$$-\sum_{i=1}^{[n/_2]+1} 2^{2i-1} \leqslant a,\ b \leqslant \sum_{i=0}^{n/2} (-2)^{2i}.$$

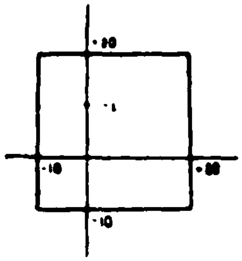

Fig. 36.

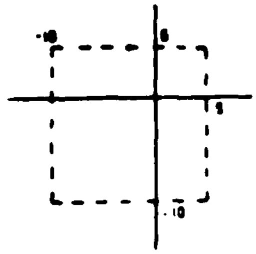

Fig. 37

Fig. 36. Range < · | $_{-2^4-1}$, −(2 − +8) < a, b < 4+16.

Fig. 37. Range < · | $_{-2^3-1}$, −(2 − +8) < a, b < 1+4.

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Are given below the tables of addition and multiplication of deductions on mod (-2) taking into account overflow. Deductions are coded, according to Table (6.5):

:	0	1	i	$1+i$
Code	0	1	2	3

In order not to confuse symbols 2; 3 as the code designations of values i, 1+i with numerical values 2; 3, we will write/record the latter in the parenthesis.

We have:

$$(2) = 1\cdot p^2 + 1p = 110 \qquad (-1) = 1\cdot p + 1 = 11$$
$$(2i) = 2\cdot p^2 + 2p = 220 \qquad (2) + i = 1\cdot p^2 + 1p + 2 = 112$$
$$(2) + (2i) = 3\cdot p^2 + 3p = 330 \qquad -1 + i = 1\cdot p + 3 = 13.$$
$$(1) + (2i) = 2\cdot p^2 + 2p + 1 = 221$$

(1) Таблица сложения

+	0	1	2	3
0	0	1	2	3
1	1	110	3	112
2	2	3	220	221
3	3	112	221	330

(2) Таблица умножения

×	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	2	11	13
3	0	3	13	220

Key: (1). Table of addition. (2). Multiplication table.

Observation. The emphasized digits in the tables are the values of overflow.

In the binary notation these overflows take the form:

(1) Аддитивные переполнения	(2) Мультипликативные переполнения
11 = 0101	1 = 01
22 = 1010	22 = 1010
33 = 1111	

Key: (1). Additive overflows. (2). Multiplicative overflows.

The chart technology part of the realization of additive overflow in the binary version on the even and odd bits is uniform; the nature of multiplicative overflow on the even and odd bases/bases is different.

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The given tables make it possible by the ordinary methods of positional arithmetic to construct the algorithms of addition and multiplication.

Let us note that in the case of the quaternary coding of deductions on mod (-2) useful are with the execution of addition the following zero cycles:

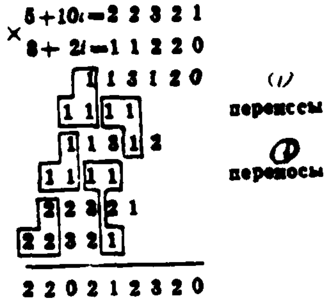
$$
+\begin{array}{r} -1=11 \\ +1=\ 1 \\ \hline 0=00 \end{array}
\qquad
+\begin{array}{r} -i=22 \\ +i=\ 2 \\ \hline 0=00 \end{array}
\qquad
\begin{array}{r} -1-i=33 \\ 1+i=\ 3 \\ \hline 0\ =\ 00 \end{array}
\qquad
\begin{array}{c} 33 \\ 2 \\ \hline 11 \end{array}
\qquad
\begin{array}{c} 33 \\ 1 \\ \hline 22 \end{array}
$$

Examples. To sum the numbers:

$$
\begin{array}{rcccccc}
5+10i= & 2 & 2 & 3 & \boxed{2} & 1 \\
8+2i= & 1 & 1 & \boxed{2} & \boxed{2} & 0 \\
\hline
 & 3 & 3 & 3 & 0 & 1
\end{array}
\qquad
\begin{array}{rcccccccl}
6+25i= & 2 & 2 & 1 & 3 & 0 & 1 & 2 & \\
5\ 2+37i= & 3 & 3 & 1 & 0 & 3 & 0 & 2 & \\
\hline
 & \boxed{1} & \boxed{1} & 0 & \boxed{3} & \boxed{3} & 1 & 0 & \\
 1 & \boxed{1} & & & & \boxed{2} & 2 & & \text{I перенос}^{(1)} \\
 \boxed{2} & 2 & & & & & & & \text{II перенос} \\
 \boxed{2\ 2} & & & & & & & & \text{III перенос} \\
\hline
 3 & 0 & & 0 & 1 & 1 & 3 & 0 &
\end{array}
$$

Key: (1) transfer.

To multiply the numbers:



Key: (1) transfers.

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Chapter 5.

NONPOSITIONAL NUMERATION SYSTEMS WITH THE COMPOSITE BASES/BASES.

§ 1. Essence of the nonpositional numeration systems.

The nonpositional numeration systems arose in connection with the development of the methods of the deparallelization of operations at the level of arithmetic operations. For such systems are characteristic many bases/bases. Code words are divided/marked off into the independent components on all bases/bases. The process of the transformation of information is decomposed/expanded into the stages of maximum length, for each of which processing code words is reduced to the independent processing of all components. Each such stage of the transformation of information we will call modular, and the stages of transformation of information, on which appears the need for analyzing one or the other set of the components of code words, by nonmodule.

It is logical that the tendency to reduce the number of nonmodule/nonmondulus stages thereby to increase the depth of deparallelization leads to an increase in the equipment. The compromise solution of these contradictory tendencies depends on many factors, most important of which they are:

a) the field of the tasks for solving which are intended the projected/designed computational means;

b) element basis;

c) the technical and economic indices of computational means.

At the present time wide reputation obtained the nonpositional numeration system in residual classes [1, 2, 11, 12, 13, 14, 15].

In this chapter is stated certain development of the theory of the nonpositional numeration systems in the residual classes with the composite bases/bases, initiated in monograph [1].

The idea of the nonpositional numeration system in the residual classes rests on the remainder theorem (theorem 3.3.3).

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Let $P_n = p_1 p_2 \cdots p_n$, where $p_1, p_2, \ldots, p_n$ — pair-wise mutually simple of ts.k.ch. Let us introduce the designation

$$P_{kn} = p_1 p_2 \cdots p_{k-1} p_{k+1} \cdots p_n.$$

In accordance with theorem 3.33 set ts.k.ch. of the form

$$\sum_{k=1}^{n} \zeta_k B_k,$$

where

$$B_k = < P_{kn}^{-1} \mid_{p_k} \cdot P_{kn}, \qquad (1.1)$$

and ζ_k passes values p.s.v. on mod p_k $(k=1, 2, \ldots, n)$, are formed p.s.v. ζ_n on composite/compound modulus/module P_n.

The set of numbers $\{B_k\}$ is called the orthogonal base of the nonpositional numeration system. Any ts.k.ch. $z \in \zeta_n$ is uniquely determined by the set of its deductions on moduli/modules $p_1, p_2, \ldots, p_n$:

$$z \div (\zeta_1, \zeta_2, \ldots, \zeta_n).$$

This the vector recording of number z determines the so-called nonpositional code of number z. It is assumed that the sequence of bases/bases is regulated somehow, in that case of k-th of the component of this recording is called k-th the bit of the nonpositional code.

For any ts.k.ch. $z_1, z_2 \in < \cdot \mid_{P_n} = \zeta_n$ are valid the relationships/ratios

$$< z_1 + z_2 \mid_{p_n} = \sum_{k=1}^{n} < \zeta_k^{(1)} + \zeta_k^{(2)} \mid_{p_k} B_k,$$
$$< z_1 \cdot z_2 \mid_{p_n} = \sum_{k=1}^{n} < \zeta_k^{(1)} \cdot \zeta_k^{(2)} \mid_{p_k} B_k, \qquad (1.2)$$

where

$$z_1 = \sum_{k=1}^{n} \zeta_k^{(1)} B_k,$$
$$z_2 = \sum_{k=1}^{n} \zeta_k^{(2)} B_k.$$

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In other words, the modular operations above the elements/cells p.s.v. σ_n on composite/compound modulus/module P_n are realized by means of the parallel and independent execution of the corresponding modular operations on the deductions of operands z_1 and z_2 on bases/bases $p_1, p_2, \ldots, p_n$:

$$< z_1 + z_2 \mid_p = (< \zeta_1^{(1)} + \zeta_1^{(2)} \mid_{p_1}, < \zeta_2^{(1)} + \zeta_2^{(2)} \mid_{p_2}, \ldots$$
$$\ldots, < \zeta_n^{(1)} + \zeta_n^{(2)} \mid_{p_n}),$$
$$< z_1 \cdot z_2 \mid_p = (< \zeta_1^{(1)} \cdot \zeta_1^{(2)} \mid_{p_1}, < \zeta_2^{(1)} \cdot \zeta_2^{(2)} \mid_{p_2}, \ldots$$
$$\ldots, < \zeta_n^{(1)} \cdot \zeta_n^{(2)} \mid_{p_n}).$$

Let now σ_n' — any other p.s.v. on $\mathrm{mod}\, P_n$. Arbitrary ts.k.ch. $w \in \sigma_n'$ in a unique manner is represented in the form

$$w = z + rP_n, \qquad (1.3)$$

where $z \in \sigma_n$.

From the latter/last equality it follows that

$$<w \mid_{p_k} = <z \mid_{p_k} = \zeta_k.$$

Taking into account equality (1.1), relationship/ratio (1.3) can be presented in the form

$$w = \sum_{k=1}^{n} \zeta_k \cdot B_k + r_w P_n. \qquad (1.4)$$

Equality (1.4) determines the nonpositional representation of element/cell $w \in \sigma_n$; ts.k.ch. $p_1, p_2, \ldots, p_n$ they are called the bases/bases of this representation.

Value r_w we will call the rank of element/cell $w \in \sigma_n'$. It is clear that rank r_w depends on the method of assignment of p.s.v. σ_n'. With that fixed/recorded p.s.v. σ_n' rank r_w is the function of number w. The value of rank it is possible to affect by some modifications of nonpositional representation of numbers.

Thus, in particular, let us note that together with representation (1.4) it is possible to use also the following representation. Noting that

$$B_k = <P_{kn}^{-1} \mid_{p_k} \cdot P_{kn},$$

let us isolate "whole part" of the division of the product

$$\zeta_k \cdot <P_{kn}^{-1} \mid_{p_k}$$

on p_k.

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We will obtain

$$\zeta_k \cdot < P_{kn}^{-1} \mid_{p_k} = \beta_{kn} + x_k p_k, \qquad (1.5)$$

where

$$\beta_{kn} = < \zeta_k \cdot < P_{kn}^{-1} \mid_{p_n} \mid_{p_k}. \qquad (1.6)$$

Substituting (1.5) and (1.4), we will obtain

$$w = \sum_{k=1}^{n} \beta_{kn} P_{kn} + \nu_w P_n, \qquad (1.7)$$

where

$$\nu_w = r_w + \sum_{k=1}^{n} x_k. \qquad (1.8)$$

Formula (1.7), in contrast to (1.4), we will call the standardized/normalized nonpositional representation ts.k.ch. $w \in$, value β_{kn} — by the standardized/normalized deductions, and ν_w — by normalized rank.

It must be noted that the tables of the modular multiplication of the standardized/normalized deductions on mod p_k require certain transformation.

Actually/really, the direct modular multiplication of two normalized deductions gives

$$< \beta_{kn}^{(1)} \cdot \beta_{kn}^{(2)} \mid_{p_k} = < \zeta_k^{(1)} \cdot \zeta_k^{(2)} < P_{kn}^{-1} \mid_{p_k}^{2} \mid_{p_k}.$$

Consequently, in order that the output/yield of table would be obtained the standardized/normalized deduction, the modular product

of two normalized deductions must be it must normalized, i.e., to multiply to value $<P_{kn}|_{p_k}$. Thus, the elements/cells of the table of modular multiplication must be determined in accordance with the rule

$$\ll P_{kn}|_{p_k}\cdot\beta^{(1)}_{kn}\cdot\beta^{(2)}_{kn}|_{p_k}.$$

The tables of modular addition are not subject to similar changes, since

$$<\beta^{(1)}_{kn}+\beta^{(2)}_{kn}|_{p_k}=<(\zeta^{(1)}_k+\zeta^{(2)}_k)<P^{-1}_{kn}|_{p_k}|_{p_k}.$$

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Let us designate through Σ_n p.s.v. on composite/compound modulus/module P_n, that determined by numbers of form

$$\sum_{k=1}^{n}\beta_{kn}\cdot P_{kn},$$

where each β_n passes value of p.s.v. on $\bmod p_n$ respectively. It is obvious that, by force (1.3), p.s.v. σ_n and Σ_n are the unique representatives of the sets ts.k.ch., which have unranked, nonpositional representation of the form (1.4) and (1.7). Formula (1.8) expresses a change in the rank upon transfer from p.s.v. σ_n' p.s.v. Σ_n. Let us give some illustrative examples.

Let us select as the bases/bases p.k.ch. $p_1=-1+i$, $p_2=1+2i$, $p_3=3+2i$, then $P_3=(-1+i)(1+2i)(3+2i)=-7-9i$. As p.s.v. let us select p.s.a.n.v;

$$<\cdot \mid^{-}_{-1+i} = \{0,\ 1\};\quad <\cdot \mid^{-}_{1+2i} = \{0,\ 1,\ -1,\ i,\ -i\};$$

$$<\cdot \mid^{-}_{3+2i} = \{0,\ 1,\ 2,\ -2i,\ -1+i,\ i,\ 1+i,\ -1-i,$$
$$-i,\ 1-i,\ 2i,\ -2,\ -1\}.$$

Let us determine

$$< P_{13} \mid^{-}_{-1+i} = < -1+8i \mid^{-}_{-1+i} = 1,$$

$$< P_{23} \mid^{-}_{1+2i} = < -5+i \mid^{-}_{1+2i} = i,$$

$$< P_{33} \mid^{-}_{3+2i} = < -3-i \mid^{-}_{3+2i} = i,$$

$$< P_{13}^{-1} \mid^{-}_{-1+i} = < \frac{1}{-1+8i} \mid^{-}_{-1+i} = 1,$$

$$< P_{23}^{-1} \mid^{-}_{1+2i} = < \frac{1}{-5+i} \mid^{-}_{1+2i} = -i,$$

$$< P_{33}^{-1} \mid^{-}_{3+2i} = < \frac{1}{-3-i} \mid^{-}_{3+2i} = -i,$$

$$B_1 = < P_{13}^{-1} \mid^{-}_{p_1} \cdot P_{13} = -1+8i,$$

$$B_2 = < P_{23}^{-1} \mid^{-}_{p_2} \cdot P_{23} = 1+5i,$$

$$B_3 = < P_{33}^{-1} \mid^{-}_{p_3} P_{33} = -1+3i.$$

Further let us make table of the equivalency of deductions and standardized/normalized deductions on moduli/modules p_1, p_2 and p_3, for which is used formula (1.6).

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(1) Для mod(−1+i)

$\xi_\kappa = \langle \cdot \mid \overline{}_{-1+i}$	$\beta_{\kappa 1} = \langle \xi_\kappa \langle P_{13}^{-1} \mid_{-1+i} \mid_{-1+i}$
0	0
1	1

(1) Для mod(1+2i)

$\xi_\kappa = \langle \cdot \mid_{1+2i}$	$\beta_{\kappa 2} = \langle \xi_\kappa \langle P_{23}^{-1} \mid_{1+2i} \mid_{\overline{1+2i}}$
0	0
1	$-i$
i	+1
$-i$	−1
−1	i

(1) Для mod(3+2i)

$\xi_\kappa = \langle \cdot \mid_{3+2i}$	$\beta_{\kappa 3} = \langle \xi_\kappa \langle P_{33}^{-1} \mid_{\overline{3+2i}} \mid_{\overline{3+2i}}$
0	0
1	$-i$
2	$-2i$
$-2i$	−2
$-1+i$	$1+i$
i	1
$1+i$	$1-i$
$-i$	−1
$1-i$	$-1-i$
$2i$	2
−2	$2i$
−1	i
$-1-i$	$-1+i$

Key: (1). For.

Let us make table of modular multiplication of the standardized/normalized deductions in accordance with rule $\langle\langle P_{k3} \mid \overline{}_{p_k} \cdot \beta_{k3}^{(1)} \cdot \beta_{k3}^{(2)} \mid_{p_k}$.

(1) Для $\bmod(-1+i)$, $<P_{13} \mid \overline{p_1} = 1$

β_{F3} \ β_{K3}	0	1
0	0	0
1	0	1

(1) Для $\bmod(1+2i)$, $<P_{23} \mid \overline{p_1} = i$

β_{K3} \ β_{K3}	0	$-i$	1	-1	i
0	0	0	0	0	0
$-i$	0	$-i$	1	-1	i
1	0	1	i	$-i$	-1
-1	0	-1	$-i$	i	1
$-i$	0	i	-1	1	$-i$

Key: (1). For.

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(1) Для mod$(3+2i)$, $\langle P_{33} \mid^{-}_{p_s} = i$

$P_{на}$ \ $P_{ка}$	0	$-1-i$	1	$-i$	2	$1-i$	$-2i$	$-1+i$	-2	i	-1	$1+i$	$2i$
0	0	0	0	0	0	0	0	0	0	0	0	0	0
$-1-i$	0	-2	$1-i$	$-1-i$	i	$-2i$	1	$2i$	$-i$	$1+i$	$-1+i$	2	-1
1	0	$1-i$	i	1	$2i$	$1+i$	2	$-1-i$	$-2i$	-1	$-i$	$-1+i$	-2
$-i$	0	$-1-i$	1	$-i$	2	$1-i$	$-2i$	$-1+i$	-2	i	-1	$1+i$	$2i$
2	0	i	$2i$	$-i$	$-1-i$	-1	$-1+i$	1	$1+i$	-2	$-2i$	$-i$	$1-i$
$1-i$	0	$-2i$	$1+i$	2	-1	2	i	-2	1	$-1+i$	$-1-i$	$2i$	$-i$
$-2i$	0	1	2	$1-i$	$-1+i$	i	$1-i$	$-i$	$1-i$	$2i$	-2	1	$-1-i$
$-1+i$	0	$2i$	$-1-i$	$-2i$	1	-2	$-i$	2	1	$1-i$	$1+i$	$-2i$	i
-2	0	$-i$	$-2i$	$-1+i$	$1+i$	1	$1-i$	-1	$-1+i$	2	$2i$	i	$-1+i$
i	0	$1+i$	-1	-2	2	$-1+i$	$2i$	$1-i$	2	$-i$	1	$-1-i$	$-2i$
-1	0	$-1+i$	$-i$	i	$-2i$	$-1-i$	-2	$1+i$	$2i$	1	i	$1-i$	2
$1+i$	0	2	$-1+i$	-1	$-i$	$2i$	1	$-2i$	i	$-1-i$	$1-i$	-2	1
$2i$	0	-1	-2	$1+i$	$1-i$	$-i$	$-1-i$	i	$-1+i$	$-2i$	2	1	$1-i$

Key: (1). For.

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Example 1. Let us present ts.k.ch. -1+7i and -2-5i in the nonpositional code. Their a.n.v. modulo -1+i, 1+2i, and 3+2i will be:

$$\langle -1+7i \mid^{-}_{-1+i} = 0,\ \langle -1+7i \mid^{-}_{1+2i} = -i,\ \langle -1+7i \mid^{-}_{3+2i} = -i,$$

$$\langle -2-5i \mid^{-}_{-1+i} = 1,\ \langle -2-5i \mid^{-}_{1+2i} = -i,\ \langle -2-5i \mid^{-}_{3+2i} = -1.$$

therefore

$$-1+7i = \xi_1^{(1)} B_1 + \xi_2^{(1)} B_2 + \xi_3^{(1)} B_3 + r_1 P_3 = 0\cdot(-1+8i) +$$
$$+ (-i)(1+5i) + (-i)(-1+3i) + (-i)(-7-9i);$$
$$-2-5i = \xi_1^{(2)} B_1 + \xi_2^{(2)} B_2 + \xi_3^{(2)} B_3 + r_2 P_3 = 1\cdot(-1+8i) +$$
$$+ (-1)(1+5i) + (i)(-1+3i) + (-1)(-7-9i).$$

Example 2. For the standardized/normalized nonpositional representation deductions $\beta_k^{(j)}$ according to the table of equivalency we convert into standardized/normalized deductions $\beta_{kn}^{(j)}$. For ts.k.ch. $-1+7i$:

$$0 \sim 0$$
$$-i \sim -1$$
$$-i \sim -1$$

for ts.k.ch. $-2-5i$:

$$1 \sim 1$$
$$-i \sim -1$$
$$-1 \sim i$$

Therefore standardized/normalized nonpositional representation takes the form

$$-1+7i = \beta_{13}^{(1)} \cdot P_{.3} + \beta_{23}^{(1)} \cdot P_{23} + \beta_{33}^{(2)} \cdot P_{33} + \gamma_1 P_3 = 0 \cdot (-1+8i) +$$
$$+ (-1)(-5+i) + (-1)(-3-i) + (-i)(-7-9i);$$
$$-2-5i = \beta_{13}^{(2)} \cdot P_{.3} + \beta_{23}^{(2)} \cdot P_{23} + \beta_{33}^{(2)} \cdot P_{33} + \gamma_2 P_3 = 1 \cdot (-1+8i) +$$
$$+ (-1)(-5+i) + (i)(-3-i) + 1(-7-9i).$$

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Example 3. Let us find the sum of the numbers $-1+7i$ and $-2-5i$, preset in the nonpositional code:

$$\begin{array}{r} -1+7i+(0,\,-i,\,-i) \\ + \underline{-2-5i+(1,\,-i,\,-1)} \\ -3+2i+(<0+1\mid_{\overline{1+i}},\ <-i+(-i)\mid_{\overline{1+2i}},\ <-i+(-1)\mid_{\overline{3+2i}}) \end{array}$$

or

$$-3+2i+(1,\,1,\,-1-i).$$

Example 4. Let us find the product of the numbers 1+i and 1+4i, preset in the standardized/normalized deductions:

$$1+i+(0,\,-1,\,1-i),$$
$$1+4i+(1,\,i,\,-1).$$

For obtaining the product we will use the table of modular multiplication in the standardized/normalized deductions

$$(1+i)\cdot(1+4i)+(<0\cdot1\mid_{\overline{1+i}},\ <-1\cdot i\mid_{\overline{1+2i}},\ <(1-i)\,(-1)\mid_{\overline{3+2i}}),$$
$$(1+i)\,(1+4i)=-3+5i=(0,\,1,\,-1-i).$$

and actually -3+5i has the nonpositional standardized/normalized representation:

$$-3+5i=0\cdot P_{13}+1\cdot P_{23}+(-1-i)\,P_{33}+\gamma P_3=0\cdot(-1+8i)+$$
$$+1\cdot(-5+i)+(-1-i)\,(-3-i)+0\cdot P_3.$$

§ 2. Selection of the range of the nonpositional numeration system.

As the range of the nonpositional numeration system with composite bases/bases $p_1, p_2, \ldots, p_n$ is selected certain p.s.v. on mod $P_n (P_n = p_1 p_2 \ldots p_n)$. Selection of p.s.v. can be dictated by different considerations. Thus, if we proceed from the requirement of unranked nonpositional representation of numbers, then as p.s.v. on mod P can be chosen set $\sum_{n}$, determined above.

In this case overflow for the range with the addition ts.k.ch., belonging to range Σ_m, is determined by the sum of the overflows on each bit of the nonpositional code of operands.

This range can be interesting for solving the questions of detection and correction of errors during the transmission of planar (complex-valued) information, and also with the solution of other special problems of the theory of complex numbers. However, during the construction of the arithmetic numeration system with the range of numbers of the mentioned type significantly is complicated the operation of multiplication, in this case the geometric image of range carries the specific character (strong vacuity), which creates restriction for developing the general-purpose arithmetic numeration system.

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From this point of view the most acceptable selection of range is p.s.v. according to mod P_n, described by the mixed positional representation

$$\zeta_1 + \zeta_2 p_1 + \zeta_3 p_1 p_2 + \ldots + \zeta_n p_1 \cdots p_{n-1},$$

where variable/alternating ζ_k pass value p.s.v. on $\bmod p_k$ (k=1, 2, ..., n). As already it showed, the geometric image of such p.s.v.

depends not only on the selection of bases/bases $p_1, p_2, \ldots, p_n$, but also on the method of assignment of p.s.v. with respect to modulus/module p_k. Certain print to the algorithms of arithmetic operations in the nonpositional numeration systems, which are based on similar ranges of numbers, superimposes absence in them in general the property of the invariance (see Chapter 3, § 3).

In connection with this the ranges, determined by the mixed positional code, it is expedient to designate by symbol where together with the indication of base$(p_1, p_2, \ldots, p_n)$ is indicated also the order of their sequence.

Apparently, most convenient range for developing the machine arithmetic of complex numbers, which has wide applied value, is totality of ts.k.ch., included in the square whose sides are parallel to the axes of the rectangular coordinate system of the composite plane z. Similar unique representatives of p.s.v. they are p.s.v. on the real moduli/modules. Thus, as rice it would be possible to select the numbers

$$p_1 = 2,\ p_2 = 3,\ p_3 = 5,\ p_4 = 7,\ p_5 = 11,\ p_6 = 13.$$

However, for bases/bases 7, 11, 13, ... the set of the elements/cells, which correspond to p.s.v., is determined by quantity 49, 121, 169, ... A rapid increase in the number of deductions substantially impedes the tabular realization of modular operations.

Thus, on mod 13 tables of modular operations they will contain 169x169 elements/cells. If a similar multiplication table is shortened 16 times due to the special coding (Chapter 3, § 4), then, nevertheless, the abbreviated/reduced table will remain sufficiently bulky. Departure from this position will give the selection of the pair-wise conjugated/combined bases/bases. It is known that if ts.k.ch. a+bi satisfies condition (a, b)=1, then numbers a+bi and a-bi are mutually simple.

Taking into account this observation it is possible to select the pair-wise conjugated/combined bases/bases so as not to break the basic requirement, presented to the basis of the nonpositional numeration system in the residual classes, namely: pairwise mutual simplicity of bases/bases.

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This requirement, as is known, ensures the unique representation of deduction according to the composite/compound modulus/module with the set of remainders/residues (deductions) on the pair-wise mutually simple moduli/modules whose product is equal to the preset composite/compound modulus/module.

A similar type as bases/bases can serve, for example, numbers

2+-3i, 1+-4i, 3+-4i, 2+-5i, 1+-6i, 4+-5i, ... and so forth.

§ 3. Nonpositional numeration system with the pair-wise conjugated/combined bases/bases.

Let be preset the pair-wise mutually simple bases/bases

$$p_1, \bar{p}_1;\ p_2, \bar{p}_2;\ p_3, \bar{p}_3; \dots;\ p_n, \bar{p}_n. \qquad (3.1)$$

Let us introduce designations for the norms of the bases/bases:

$$q_k = \| p_k \| = p_k \cdot \bar{p}_k,$$

$$Q_n = q_1 \cdot q_2 \dots q_n.$$

Range as p.s.v. on $\bmod Q_n$, that determined by the mixed positional representation, can be preset doubly.

First method. Range $\sigma_{2n}^{\pm} = (p_1 \bar{p}_1 p_2 \bar{p}_2 \dots p_n \bar{p}_n$

$$w \in \sigma_{2n}^{\pm},$$

if

$$w = \xi_1 + \eta_1 \cdot p_1 + \xi_2 \cdot p_1 \cdot \bar{p}_1 + \eta_2 \cdot p_1 \cdot \bar{p}_1 \cdot p_2 + \dots +$$
$$+ \xi_n \cdot p_1 \cdot \bar{p}_1 \dots p_{n-1} \cdot \bar{p}_{n-1} + \eta_n \cdot p_1 \cdot \bar{p}_1 \dots p_{n-1} \cdot \bar{p}_{n-1} \cdot p_n =$$
$$= \sum_{k=1}^{n} \xi_k q_1 \dots q_{k-1} + \sum_{k=1}^{n} \eta_k q_1 \dots q_{k-1} \cdot p_k,$$

where $\xi_k \in \langle \cdot |_{\bar{p}_k}^{\pm}, \ \eta_k \in \langle \cdot |_{p_k}^{\pm} \ (1 \leq k \leq n)$

(here and below symbol $\langle \cdot |_{p_k}^{\pm}$ it means that it is intended p.s.n.v. or p.s.a.n.v.).

Second method. Range $\sum_{2n}^{\pm}=(q_1 \cdot q_2 \ldots q_n)$.

$$z \in \sum_{2n}^{\pm},$$

if

$$z=\zeta_1+\zeta_2 q_1+\zeta_3 q_1 q_2+\ldots+\zeta_n q_1 q_2 \ldots q_{n-1}, \qquad (3.2)$$

where

$$\zeta_k \in <\cdot \mid_{q_k}^{\pm} \quad (1 \leqslant k \leqslant n). \qquad (3.3)$$

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Theorem 3.3.4 makes it possible to establish/install the connection between these two ranges. According to the mentioned theorem,

$$\zeta_k=\xi_k+\eta_k \cdot p_k+\varepsilon_k \cdot q_k,$$

where

$$\xi_k \in <\cdot \mid_{p_k}^{\pm}, \; \eta_k \in <\cdot \mid_{p_k}^{\pm},$$

$$\| \varepsilon_k \| = \begin{cases} 0 \\ 1 \end{cases}.$$

Consequently,

$$z=\sum_{k=1}^{n} \zeta_k q_1 \ldots q_{k-1}=\sum_{k=1}^{n} \xi_k q_1 \ldots q_{k-1}+\sum_{k=1}^{n} \eta_k q_1 \ldots q_{k-1} p_k+$$

$$+\sum_{k=1}^{n} \varepsilon_k q_1 \ldots q_k,$$

i.e.

$$z=w+\sum_{k=1}^{n} \varepsilon_k q_1 \ldots q_k.$$

From the same theorem it follows that range s_{2n} is not invariant to the transfer of bases/bases, the set of points having sufficiently complex geometric configuration.

Theorem 3.1. Range $\Sigma^{\pm}_{2n} = (q_1 \cdot q_2 \cdots q_n)$ is invariant to the arbitrary transfer of any pairs of the compositely conjugated/combined bases/bases, i.e.

$$(q_1 q_2 \ldots q_n) = (q_{i_1} q_{i_2} \ldots q_{i_n}).$$

In this case the set of points $\Sigma^{\pm}_{2n}$ is represented as square p.s.v. $<\cdot|^{\pm}_{Q_n}$.

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Proof. So that $\zeta_k = x_k + iy_k \in <\cdot|^{\pm}_{q_k}$, is necessary and it is sufficient so that would be satisfied the condition

$$x_k, y_k \in |\cdot|^{\pm}_{q_k},$$

then for any ts.k.ch. $z \in \Sigma^{\pm}_{2n}$:

$$z = u + iv = \sum_{k=1}^{n} \zeta_k q_1 \ldots q_{k-1} = \sum_{k=1}^{n} x_k q_1 \ldots q_{k-1} + + i \sum_{k=1}^{n} y_k q_1 \ldots q_{k-1}$$

is carried out

$$u, v \in |\cdot|^{\pm}_{Q_n}.$$

Consequently, set $\Sigma^{\pm}_{2n}$ is described respectively by square p.s.a.n.v. or p.s.n.v. on real modulus/module Q_n.

Invariance $\Sigma^{\pm}_{2n}$ follows from the fact that for the arbitrary

transfer of bases/bases $q_{i_1}, q_{i_2}, \ldots, q_{i_n}$ are valid inequality

$$\left\{\begin{matrix} 0 \\ -\frac{Q_n}{2} \end{matrix}\right. \leqslant \sum_{k=1}^{n} x_{ik} q_{i1} \ldots q_{ik-1} < \left\{\begin{matrix} Q_n \\ \frac{Q_n}{2} \end{matrix}\right.,$$

$$\left\{\begin{matrix} 0 \\ -\frac{Q_n}{2} \end{matrix}\right. \leqslant \sum_{k=1}^{n} y_{ik} q_{i1} \ldots q_{ik-1} < \left\{\begin{matrix} Q_n \\ \frac{Q_n}{2} \end{matrix}\right.,$$

where $x_{ik}, y_{ik} \in | \cdot |^{\pm}_{q_{ik}}$.

Let us show the validity of these inequalities in the case of p.s.n.v.

Since

$$0 \leqslant x_{ik} < q_{ik},$$

that

$$0 \leqslant \sum_{k=1}^{n} x_{ik} q_{i1} \ldots q_{ik-1} \leqslant \sum_{k=1}^{n} (q_{ik} - 1) q_{i1} \ldots q_{ik} =$$

$$= \sum_{k=1}^{n} q_{i1} \ldots q_{ik} - \sum_{k=1}^{n} q_{i1} \ldots q_{ik-1} = Q_n - 1.$$

For the case of p.s.a.n.v. the validity of inequalities is checked analogously.

The invariance of range acquires important value for the machine arithmetic in connection with the following property of such systems of numbers.

Let us introduce the designations:

1. I= { 1, 2, ..., n } - the set of indices with the preset sequence of basas/bases $q_1, q_2, \ldots, q_n$; $I' = \{i_1, i_2, \ldots, i_n\}$ — certain arbitrary transfer of elements of set I.

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2. $J = \{j_1, j_2, \ldots, j_r\}$ — certain subset of set I.

3. (Q_J) — p.s.v. on mod Q_J, where $Q_J = q_{j_1} q_{j_2} \ldots q_{j_r}$, determined by set of numbers

$$\zeta_{j_1} + \zeta_{j_2} q_{j_1} + \zeta_{j_3} q_{j_1} q_{j_2} + \ldots + \zeta_{j_r} q_{j_1} \ldots q_{j(r-1)}$$

$$\zeta_{j_k} \in \langle \cdot \mid \tilde{q}_{j_k}$$

Theorem 3.2. P.s.v. (Q_I), is invariant to the transfer of bases, it possesses the following property:

any ts.k.ch. $z \in (Q_I)$ is unambiguously represented in the form

$$z = u + vQ_J, \qquad (3.4)$$

where

$$u \in (Q_J), \quad v \in (Q_{I-J}), \quad (J \subset I)$$

and, conversely, any ts.k.ch. z, represented in the form

$$u + vQ_J,$$

where

$$u \in (Q_J), \; v \in (Q_{I-J}),$$

it belongs to set (Q_I).

Proof. Let $J=\{j_1, j_2, \ldots, j_n\} \in I$ and $I'=\{j_1, j_2, \ldots, j_r, k_1, k_2, \ldots, k_{n-m}\}$.

Since $(Q_I)=(Q_{I'})$, that for any ts.k.ch.z $z \in (Q_{I'})$ we have

$$z=\zeta_{j_1}+\zeta_{j_2} q_{j_1}+\ldots+\zeta_{j_r} q_{j_1} \ldots q_{j_{r-1}}+Q_I(\zeta_{k_1}+\zeta_{k_2} q_{k_1}+\ldots+$$
$$+\zeta_{k_{n-m}} q_{k_1} q_{k_2} \ldots q_{k_{n-m-1}})=u+vQ_I,$$

where, according to the determination

$$u=\zeta_{j_1}+\zeta_{j_2} q_{j_1}+\ldots+\zeta_{j_r} q_{j_1} \ldots q_{j_{r-1}} \in (Q_J), \qquad (3.5)$$
$$v=\zeta_{k_1}+\zeta_{k_2} q_{k_1}+\ldots+\zeta_{k_{n-m}} q_{k_1} \ldots q_{k_{n-m-1}} \in (Q_{I-J}). \quad (3.6)$$

of uniqueness of this representation follows from the uniqueness of the mixed positional representation of numbers

Conversely, $z=u+vQ_J$, where $u \in (Q_J)$, $v \in (Q_{I-J})$. In accordance with the determination p.s.v. of form (Q_J) ts.k.ch. it is unambiguously represented by relationships/ratios of type (3.5) and (3.6). Consequently,

$$z \in (Q_{I'}).$$

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But according to condition $(Q_{I'})=(Q_I)$, therefore

$$z \in (Q_I).$$

Corollary 1. let it be in the form (3.4) ts.k.ch. it is fixed and

$$< u \mid p_{j_k} = a_{j_k} \quad (j_k \in J),$$

v passes the value of range (Q_{I-J}). then as a result form (3.4) describes the set ts.k.ch. of range (Q_I), of those united under the sign/criterion:

these deductions of ts.k.ch. on each basis/base $p_{j_k} (j_k \in J)$ are equal to each other and are determined by value a_{j_k}.

Corollary 2. If $z \in (Q_I)$ and $z = z' Q_{I-J}$, then $z' \in (Q_J)$, and, conversely, if

$$z' \in (Q_J), \text{ then } z' Q_{I-J} \in Q_I.$$

In the connection and discussed, of two p.s.v. σ_{2n} and Σ_{2n} it is expedient to select as the range of set ts.k.ch. Σ_{2n}.

Thus, subsequently under the range, designated by symbol (Q_n), of the nonpositional numeration system with pair-wise conjugated/combined bases/bases (3.1) will be understood the set ts.k.ch. of form (3.2), where ζ_k are subordinated to condition (3.3).

Observation. The structure of range (Q_n), it is obvious, it will not be changed, if in the composition of bases/bases (3.1) will be

represanted also real bases/bases.

Let us now move on to the nonpositional representation of numbers of range (Q_n). We form the orthogonal base of the system of bases (3.1).

Let us introduce the designation

$$P_{kn}=\frac{Q_n}{p_k}. \qquad (3.7)$$

Since Q_n — real number, then

$$\overline{P}_{kn}=\frac{Q_n}{\overline{p}_k}.$$

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Taking into account $\overline{<A|_p}=<\overline{A}|_{\overline{p}}$, we note that the orthogonal base will consist of pairs of complex conjugate numbers B_k, $\overline{B}_k$:

$$B_k=m_{kn}P_{kn}, \qquad (3.8)$$

where

$$m_{kn}=<P_{kn}^{-1}|^{\pm}_{p_k}$$

and

$$\overline{B}_k=m_{kn}\overline{P}_{kn}.$$

Let $z\in(Q_n)$ and

$$<z|^{\pm}_{p_k}=\alpha_k,\ <z|^{\pm}_{\overline{p}_k}=\beta_k,$$

then

$$\sum_{k=1}^{n}(\alpha_k B_k+\beta_k\overline{B}_k)\equiv z\,(\mathrm{mod}\,Q_n).$$

Consequently,

$$z=\sum_{k=1}^{n}(\alpha_k B_k+\beta_k \bar{B}_k)+\rho_z Q_n, \qquad (3.9)$$

It is clear that rank ρ_z is equal to 0 when and only when

$$\sum_{k=1}^{n}(\alpha_k B_k+\beta_k \bar{B}_k)\in Q_n.$$

Relationship/ratio (3.9) determines the basic formula of the nonpositional representation ts.k.ch. of range (Q_n).

§ 4. Rank and its properties.

Rank as the function of number z is the most important function of the nonpositional representation of numbers, since he in conjunction with the remainders/residues determines the value of number z.

In connection with this it is interesting to study the properties of function

The symmetry of range (Q_n) is reflected also in the properties of the symmetry of rank. More graphic this is exhibited in the case of p.s.a.n.v.

Thus, using symmetry of deductions, it is not difficult to check the validity of the properties

1. $\underline{\rho(iz)} = i\rho(z)$;
2. $\overline{\rho(z)} = \rho(\overline{z})$;
3. $\underline{\rho(z)} = \rho(\underline{z})$.

Observation 1. The first property assumes that all bases/bases p_k — odd complex numbers; the presence of even (or semieven) basis/base it will lead to the need for the account of the edge/boundary deductions on this basis/base, which will somewhat distort the structure of symmetry, expressed by the first property.

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Observation 2. The same properties of symmetry in accordance with the principle of isomorphism p.s.v. on assigned modulus/module (theorem 2.3.2) can be spread also to the case of p.s.n.v.

Subsequently let us pause at the study of the properties of the standardized/normalized rank. On the basis of representation (3.9) and noting that

$$\alpha_k m_{kn} = \delta_{kn} + \varkappa_k p_k, \qquad (4.1)$$

$$\beta_k \overline{m}_{kn} = \gamma_{kn} + \varkappa_k' \overline{p}_k, \qquad (4.2)$$

where $\delta_{kn} = < \alpha_k m_{kn} \mid \frac{\pm}{p_k}$, $\gamma_{kn} = < \beta_k \overline{m}_{kn} \mid \frac{\pm}{\overline{p}_k}$, we will obtain the standardized/normalized nonpositional representation of number

$$z = \sum_{k=1}^{n} (\delta_{kn} P_{kn} + \gamma_{kn} \overline{P}_{kn}) + \mu(z) Q_n, \qquad (4.3)$$

where

$$\mu(z) = \rho(z) + \sum_{k=1}^{n} (\varkappa_k + \varkappa_k'). \qquad (4.4)$$

Since $P_{kn} = \bar{p}_k q_1 q_2 \ldots q_{k-1} q_{k+1} \ldots q_n$, that is useful to introduce one additional designation

$$Q_{kn} = q_1 \ldots q_{k-1} q_{k+1} \ldots q_n. \quad (4.5)$$

Then representation (4.3) is registered in the form

$$z = \sum_{k=1}^{n} (\delta_{kn} \bar{p}_k + \gamma_{kn} p_k) Q_{kn} + \mu(z) Q_n. \quad (4.6)$$

Let

$$\delta_{kn} \bar{p}_k + \gamma_{kn} p_k = w_k + v_k q_k, \quad (4.7)$$

where

$$w_k \in < \cdot \mid \frac{\pm}{q_k}.$$

Let us note here the connection of deductions ts.k.ch. w_k with the deductions ts.k.ch. z:

$$< w_k \mid \frac{\pm}{\bar{p}_k} = < zQ_{\bar{k}\bar{n}}^{-1} \mid \frac{\pm}{p_k},$$

$$< w_k \mid \frac{\pm}{\bar{p}_k} = < zQ_{kn}^{-1} \mid \frac{\pm}{\bar{p}_k}.$$

Actually/really,

$$< w_k \mid_{p_k} = < \delta_{kn} \bar{p}_k \mid_{p_k} = \ll a_k < P_{kn}^{-1} \mid_{p_k} \mid_{p_k} \mid_{p_k}.$$

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But $\ll P_{kn}^{-1} \mid_{p_k} \mid_{\bar{p}_k} = < Q_{kn}^{-1} \mid_{p_k}$ and $a_k = < z \mid_{p_k}$, therefore

$$< w_k \mid_{p_k} = < zQ_{kn}^{-1} \mid_{p_k}.$$

Substituting (4.7) in (4.6), we will obtain

$$z=\sum_{k=1}^{n} w_k Q_{kn}+\nu(z) Q_n, \tag{4.8}$$

where

$$\nu(z)=\mu(z)+\sum_{k=1}^{n}\nu_k.$$

Formula (4.8) expresses the canonical standardized/normalized nonpositional representation of ts.k.ch. of range (Q_n). Respectively deductions w_k we will call canonical deductions on pair-wise conjugated/combined bases/bases $p_k, \bar{p}_k$.

In connection with expansion (4.7) it is interesting to consider a question about two-modulus nonpositional represented ts.k.ch. with bases/bases p, $\overline{p}$.

As the range of such representation let us select square p.s.v. on mod $\|p\|$, i.e. $<\cdot|_{1p1}^{=}$.

As it follows from theorem 3.3.4, the mixed positional representation of numbers of this range does not possess the property of invariance. In accordance with the mentioned theorem any ts.k.ch. $w\in<\cdot|_{1p1}^{\pm}$ is unambiguously represented in the form

$$w=\xi+\eta p+\varepsilon q,\ (q=\|p\|), \tag{4.9}$$

where

$$\xi\in<\cdot|_{p}^{-},\ \eta\in<\cdot|_{p}^{=}$$

and

$$\|\varepsilon\|=\begin{cases}0\\1\end{cases}.$$

On the other hand, the nonpositional representation of this number takes the form

$$w = \delta\bar{p} + \gamma p + \nu q. \qquad (4.10)$$

Comparing these two expansions, we note that

$$\xi = \langle \delta\bar{p} \rangle_{\bar{p}}.$$

Therefore

$$\delta\bar{p} = \xi + \lambda p.$$

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Substituting latter/last disintegration in (4.10), we will obtain

$$w = \xi + (\gamma + \lambda)\, p + \nu q.$$

Let

$$\gamma + \lambda = \eta + \omega\bar{p},$$

then

$$w = \xi + \eta p + (\nu + \omega)\, q.$$

being congruent/equating latter/last disintegration with (4.9), we establish the connection between standardized/normalized rank ν of number w and its excess parameter ε in the mixed positional representation

$$\varepsilon = \nu + \omega,$$

where

$$\| \varepsilon \| = \begin{cases} 0 \\ 1 \end{cases}.$$

Hence it follows that the restoration/reduction of any of the parameters $-\varepsilon$ or ν on one known unavoidably requires some

transformations, connected with the determination of value ω.

Subsequently with the work with canonical deductions preference we will give up to representation (4.9) in view of the more complicated character of the behavior of rank ν.

Since canonical deduction w in the nonpositional numeration system is represented by its remainders/residues on moduli/modules p and $\bar{p}$ respectively, then it is necessary to work out the algorithm which would make it possible on the preset remainders/residues to determine the value of canonical deduction. This question will be examined into § 7. Already based on the example of the simplest two-modulus numeration system evidently, how complicated a character has a rank as remainder function. Is given below one of the versions of the description of rank as remainder functions based on the example of the canonical normalized nonpositional representation of ts.k.ch. Each ts.k.ch. $z\in(Q_n)$ is uniquely determined by the set of canonical deductions on the pair-wise conjugated/combined moduli/modules:

$$z \leftrightarrow (w_1, w_2, \ldots, w_n) \quad (w_k \in <\cdot \mid^{-}_{q_k}).$$

Let us present number $w_k Q_{kn}$ in the mixed positional code on bases/bases $q_1, q_2, \ldots, q_n$:

$$w_k Q_{kn} = \sum_{m=k}^{n} \omega_{km} q_1 q_2 \ldots q_{m-1}, \qquad (4.11)$$

where

$$\omega_{km} \in <\cdot \mid^{+}_{q_m} \quad (1 \leqslant m \leqslant n).$$

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Let us assume that bases/bases $q_1, q_2, \ldots, q_n$ are arranged/located in the ascending order of their values. Let us introduce the designation of deduction of ts.k.ch. z on modulus/module q_k

$$\alpha_k = < z \mid^{\pm}_{q_k}.$$

It is obvious,

$$< w_k Q_{kn} \mid^{\pm}_{q_k} = \alpha_k.$$

Therefore from (4.11) it follows that w_{kn} are the functions little more than of remainder/residue α_k. Let us determine the form of these functions.

Let us realize that numeration system consists of n+1 bases/bases $q_1, q_2, \ldots, q_n, q_{n+1}$. Then for the same number $z \in (Q_n)$ canonical deduction on $\bmod q_k$ changes its value and instead of value $w_k Q_{kn}$ we will have $w_k' Q_{k,n+1}$, where

$$w_k' < \alpha_k Q^{-1}_{k,n+1} \mid^{\pm}_{q_k}.$$

We have

$$w_k' Q_{k,n+1} = w_k' q_{n+1} Q_{k,n} = < w_k' q_{n+1} \mid^{\pm}_{q_k} Q_{k,n} +$$

$$+ x_{n+1} Q_n. \quad (4.12)$$

Since

$$w_k' Q_{k,n+1} \in (Q_{n+1})$$

and

$$< w_k' q_{n+1} \mid^{\pm}_{q_k} Q_{kn} \in (Q_n),$$

that, in view of the invariance of the mixed positional representation,

$$x_{n+1} \in \langle \cdot \mid {}^{\pm}_{q_{n+1}}.$$

We further note that

$$\langle w_k' q_{n+1} \mid {}^{\pm}_{q_k} = \langle\langle x_k Q^{-1}_{k,\, n+1} \mid {}^{\pm}_{q_k} q_{n+1} \mid {}^{\pm}_{q_k} =$$

$$= \langle x_k Q^{-1}_{kn} \mid {}^{\pm}_{q_k} = w_k.$$

Consequently, (4.12) it is possible to rewrite in the form

$$w_k' Q_{k,\, n+1} = w_k Q_{kn} + x_{n+1} q_1 \dots q_n.$$

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If we now here substitute (4.11), then we will obtain expansion ts.k.ch. $w_k' Q_{k,n+1}$ into the mixed positional code. Consequently, in designations (4.11)

$$x_{n+1} = \omega_{k,n+1}.$$

From (4.12) it follows that

$$x_{n+1} = \left[\frac{w_k' q_{n+1}}{q_k}\right]^{\pm},$$

where the brackets indicate whole syllable, which corresponds to the smallest, either non-negative or least positive residues (respectively). Therefore

$$\omega_{k,n+1} = \left[\frac{< \alpha_k Q_{k,n+1}^{-1} |_{q_k} q_{n+1}}{q_k}\right]^{\pm}.$$

In view of randomness n in general we obtain

$$\omega_{k,m} = \left[\frac{< \alpha_k Q_{k,m}^{-1} |_{q_k} q_m}{q_k}\right]^{\pm} \quad (m > k).$$

Is such form $\omega_{k,m}$ as remainder functions α_k, which let us register in the form $\omega_m(\alpha_k)$.

In accordance with (4.8) we form the sum

$$\sum_{k=1}^{n} w_k Q_{km} = \sum_{k=1}^{n} \sum_{m=k}^{n} \omega_m(\alpha_k) q_1 q_2 \dots q_{m-1} =$$

$$= \sum_{m=1}^{n} \left(\sum_{k=1}^{m} \omega_m(\alpha_k)\right) q_1 q_2 \dots q_{m-1}. \qquad (4.13)$$

Let

$$\sum_{k=1}^{m} \omega_m(\alpha_k) = \xi + \Delta_m q_m,$$

where ξ_m, Δ_m — be the remainder functions $\alpha_1, \alpha_2, \ldots, \alpha_k$, moreover

$$\xi_m = \left\langle \sum_{k=1}^{m} \omega_m(\alpha_k) \right\rangle_{q_m}^{\pm},$$

$$\Delta_m = \left[\frac{1}{q_m} \sum_{k=1}^{m} \omega_m(\alpha_k)\right]^{\pm},$$

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Let us introduce into the examination the function of overflow with the addition of two deductions of q_m:

$$\eta_m(\alpha+\beta) = \left[\frac{\alpha+\beta}{q_m}\right]^{\pm}.$$

If we in (4.13) take into account all overflows which appear with the summation over moduli/modules q_m ($m=2, 3, \ldots, n$) the expressions

$$\sum_{k=1}^{m} \omega_m(\alpha_k),$$

then as a result we will obtain disintegration into the mixed

positional code of number z and certain surplus, equal in magnitude and opposite on the sign to rank $\nu(z)$ of number z.

Thus, we will have

$$-\nu(z)=\Delta_n+\eta_n(\xi_n+\Delta_{n-1}+\eta_{n-1}(\xi_{n-1}+\Delta_{n-2}+\eta_{n-2}(\ldots))).(4.14)$$

The obtained formula expresses rank as remainder function $(\alpha_1, \alpha_2, \ldots, \alpha_n)$, since each of entering this formula variable/alternating is known remainder function.

From (4.14) follows that rank with an accuracy to the value of the value of function η_n is determined by the "linear" part

$$\Delta_n=\left[\frac{1}{q_n}\sum_{k=1}^{n}\omega_n(\alpha_k)\right]^{\pm},$$

in this case in the case of the least non-negative residue value it can take values of 0.1, i, 1+i, and in the case of the least positive residues - 0.1, i, 1+i, i, -1+i, -1, -1-i, -i, 1-i.

Another natural expression $\nu(z)$ as remainder functions is obtained directly from (4.8):

$$\nu(z)=\left[\sum_{k=1}^{n}\frac{<\alpha_k Q_{kn}^{-1}|_{q_k}}{q_k}\right]^{\pm}.$$

However, the direct numerical analysis of this formula is extremely difficult.

From the latter/last formula it follows that the standardized/normalized rank of canonical representation accepts the value from p.s.v. on mod n, i.e.,

$$v(z) \in \langle \cdot \mid \pm_{n} \cdot$$

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§5. Modular tables.

This designation/purpose paragraph auxiliary: to prepare all necessary tables which would compose basis for the illustration of described below algorithms.

In this case the task is placed somewhat wider: to describe tables in such completeness which would reflect the real situation, which appears with the work with virtually acceptable range (Q_n^-).

In connection with this as the bases/bases were chosen the following ts.k.ch.;

$\lvert p \rvert = q$		$q_1=13$		$q_2=17$		$q_3=25$	
p	$\bar{p}$	$p_1=2+3i$	$\bar{p}_1=2-3i$	$p_2=1+4i$	$\bar{p}_2=1-4i$	$p_3=3+4i$	$\bar{p}_3=3-4i$
		$q_4=29$		$q_5=37$		$q_6=41$	
		$p_4=2+5i$	$\bar{p}_4=2-5i$	$p_5=1+6i$	$\bar{p}_5=1-6i$	$p_6=4+5i$	$\bar{p}_6=4-5i$

The full/total/complete range of the numbers, described by these bases/bases, is determined by the set of numbers, equal to Q_n^2, i.e. $13^2 \cdot 17^2 \cdot 25^2 \cdot 29^2 \cdot 37^2 \cdot 41^2$. As p.s.v. on the appropriate moduli/modules they are selected p.s.a.n.v., thanks to which the arithmetic ts.k.ch. of range (Q_n) reflects "in small" all basic properties of symmetry, inherent in integral grid composite z of plane. It is possible to indicate three most important types of the tables: the table of modular addition, table of modular multiplication, table of recodings.

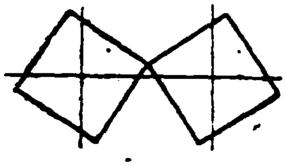

Fig. 38.

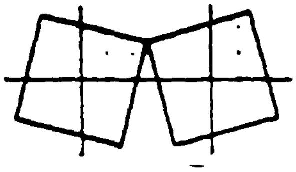

Fig. 39.

Fig. 38. Squares p.s.a.n.v. on mod(2+3i), $\rho=8$; mod(2-3i), $\rho=5$.

Fig. 39. Squares p.s.a.n.v. on mod(1+4i), $\rho=4$; mod(1-4i), $\rho=13$.

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Table of coding - single-input table, which realizes the certain function of the preset argument with the values, which belong p.s.v. on this modulus/module.

In the development of the tables of modular operations essential help proves to be theorem 2.2.5. Therefore the first group of tables consists of the tables of the isomorphic relation of composite deductions to the real deductions (straight/direct and reverse tables) (Fig. 38-43). Isomorphic relation is indicated by symbol $\leftrightarrow$. Here is indicated coding deductions, which makes it possible to

reduce the tables of modular operations.

Relative to coding of deductions it must be noted that during the engineering realization the deductions should be coded in accordance with the principle of inclusion p.s.v. on the low-order bases/bases in p.s.v. on the senior bases/bases. Therefore can be achieved/reached certain economy of equipment on the diagrams of recoding.

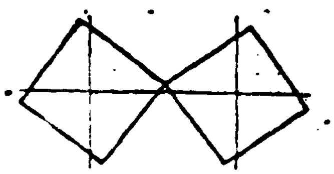

Fig. 40.

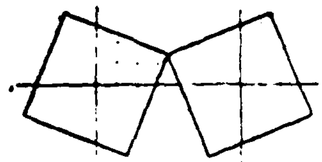

Fig. 41.

Fig. 40. Squares p.s.a.n.v. on mod(3+4i), $\rho=18$; mod(3-4i), $\rho=7$.

Fig. 41. Squares p.s.a.n.v. on mod(2+5i), $\rho=17$; mod(2-5i), $\rho=12$.

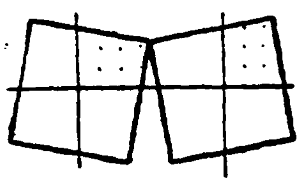

Fig. 42.

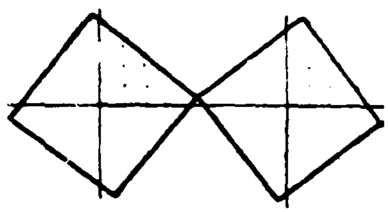

Fig. 43.

Fig. 42. Squares p.s.a.n.v. on mod(1+6i), $\rho=6$; mod(1+6i), $\rho=31$.

Fig. 43. Squares p.s.a.n.v. on mod(4+5i), on mod(4-5i), ~~$\rho=9$.~~
$\rho=32$; mod (4-5i), $\rho=9$.

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In the case in question into the basis of coding deductions is undertaken the following principle: as the mantissas of deductions are selected the deductions, located in the first quadrant of the composite plane (real axis is switched on, alleged is excluded).

The second group of tables consists of the tables of modular addition and multiplication on the appropriate moduli/modules. Tables are given in the abbreviated/reduced version in accordance with the principle of the coding of deductions accepted.

The third group of tables - table of recoding, their finding will be given in the application/appendix.

Table of the isomorphic conformity of composite and real deductions (straight lines and reverse) for the moduli/modules 2+3i, 2-3i; 1+4i, 1-4i; 3+4i, 3-4i; 2+5i, 2-5i; 1+6i, 1-6i; 4+5i, 4-5i.

$<\cdot\vert^{-}_{2+3i}\leftrightarrow$	$\vert\cdot\vert^{+}_{13}$	(1) Кодир.	$<\cdot\vert^{-}_{2-3i}\leftrightarrow$	$\vert\cdot\vert^{-}_{13}$	(1) Кодир.	$\vert\cdot\vert^{+}_{13}\leftrightarrow$	$<\cdot\vert^{-}_{2+3i}\leftrightarrow$	$<\cdot\vert^{-}_{2-3i}$
0	0	0.0	0	0	0.0	0	0	0
1	1	0.1	1	1	0.0	1	1	1
2	2	0.2	2	2	0.2	2	2	2
$1+i$	9	0.3	$1+i$	6	0.3	3	$2i$	$-2i$
i	8	1.1	i	5	1.1	4	$-1-i$	$-1+i$
$2i$	3	1.2	$2i$	10	1.2	5	$-i$	i
$-1+i$	7	1.3	$-1+i$	4	1.3	6	$1-i$	$1+i$
-1	12	2.1	-1	12	2.1	7	$-1+i$	$-1-i$
-2	11	2.2	-2	11	2.2	8	i	$-i$
$-1-i$	4	2.3	$-1-i$	7	2.3	9	$1+i$	$1-i$
$-i$	5	3.1	$-i$	8	3.1	10	$-2i$	$2i$
$-2i$	10	3.2	$-2i$	3	3.2	11	-2	-2
$1-i$	6	3.3	$-1-i$	9	3.3	12	-1	-1

$<\cdot\vert^{-}_{1+4i}\leftrightarrow$	$\vert\cdot\vert^{+}_{17}$	(1) Кодир.	$<\cdot\vert^{-}_{1-4i}\leftrightarrow$	$\vert\cdot\vert^{+}_{17}$	(1) Кодир.	$\vert\cdot\vert^{+}_{17}\leftrightarrow$	$<\cdot\vert^{-}_{1+4i}\leftrightarrow$	$<\cdot\vert^{-}_{1-4i}$
0	0	0.0	0	0	0.0	0	0	0
1	1	0.1	1	1	0.1	1	1	1
2	2	0.2	2	2	0.2	2	2	2
$1+i$	5	0.3	$1+i$	14	0.3	3	$-1+i$	$-1-i$
$2+i$	6	0.4	$1+2i$	10	0.4	4	i	$-i$
i	4	1.1	i	13	1.1	5	$1+i$	$1-i$
$2i$	8	1.2	$2i$	9	1.2	6	$2+i$	$2-i$
$-1+i$	3	1.3	$-1+i$	12	1.3	7	$-1+2i$	$-1-2i$
$-1+2i$	7	1.4	$-2+i$	11	1.4	8	$2i$	$-2i$
-1	16	2.1	-1	16	2.1	9	$-2i$	$2i$
-2	15	2.2	-2	15	2.2	10	$1-2i$	$1+2i$
$-1-i$	12	2.3	$-1-i$	3	2.3	11	$-2-i$	$-2+i$
$-2-i$	11	2.4	$-1-2i$	7	2.4	12	$-1-i$	$-1+i$
$-i$	13	3.1	$-i$	4	3.1	13	$-i$	i
$-2i$	9	3.2	$-2i$	8	3.2	14	$1-i$	$1+i$
$1-i$	14	3.3	$1-i$	5	3.3	15	-2	-2
$1-2i$	10	3.4	$2-i$	6	3.4	16	-1	-1

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$<\cdot\vert^{-}_{3+4i}$	$\leftrightarrow \vert\cdot\vert^{+}_{25}$	Кодир.	$<\cdot\vert^{-}_{3-4i}$	$\leftrightarrow \vert\cdot\vert^{+}_{25}$	Кодир.	$\vert\cdot\vert^{+}_{25}$	$\leftrightarrow <\cdot\vert^{-}_{3+4i}$	$\leftrightarrow <\cdot\vert^{-}_{3-4i}$
0	0	0.0	0	0	0.0	0	0	0
1	1	0.1	1	1	0.1	1	1	1
2	2	0.2	2	2	0.2	2	2	2
3	3	0.3	3	3	0.3	3	3	3
$1+i$	19	0.4	$1+i$	8	0.4	4	$3i$	$-3i$
$2+i$	20	0.5	$2+i$	9	0.5	5	$-2-i$	$-2+i$
$1+2i$	12	0.6	$1+2i$	15	0.6	6	$-1-i$	$-1+i$
i	18	1.1	i	7	1.1	7	$-i$	i
$2i$	11	1.2	$2i$	14	1.2	8	$1-i$	$1+i$
$3i$	4	1.3	$3i$	21	1.3	9	$2-i$	$2+i$
$-1+i$	17	1.4	$1+i$	6	1.4	10	$-1+2i$	$-1-2i$
$-1+2i$	10	1.5	$-1+2i$	13	1.5	11	$2i$	$-2i$
$-2+i$	16	1.6	$-2+i$	5	1.6	12	$1+2i$	$1-2i$
-1	24	2.1	-1	24	2.1	13	$-1-2i$	$-1+2i$
-2	23	2.2	-2	23	2.2	14	$-2i$	$2i$
-3	22	2.3	-3	22	2.3	15	$1-2i$	$1+2i$
$-1-i$	6	2.4	$-1-i$	17	2.4	16	$-2+i$	$-2-i$
$-2-i$	5	2.5	$-2-i$	16	2.5	17	$-1+i$	$-1-i$
$-1-2i$	13	2.6	$-1-2i$	10	2.6	18	i	$-i$
$-i$	7	3.1	$-i$	18	3.1	19	$1+i$	$1-i$
$-2i$	14	3.2	$-2i$	11	3.2	20	$2+i$	$2-i$
$-3i$	21	3.3	$-3i$	4	3.3	21	$-3i$	$3i$
$1-i$	8	3.4	$1-i$	19	3.4	22	-3	-3
$1-2i$	15	3.5	$1-2i$	12	3.5	23	-2	-2
$2-i$	9	3.6	$2-i$	20	3.6	24	-1	-1

$<\cdot\vert^{-}_{2+5i}$	$\leftrightarrow \vert\cdot\vert^{+}_{29}$	Кодир.	$<\cdot\vert^{-}_{2-5i}$	$\leftrightarrow \vert\cdot\vert^{+}_{25}$	Кодир.	$\vert\cdot\vert^{+}_{25}$	$\leftrightarrow <\cdot\vert^{-}_{2+5i}$	$\leftrightarrow <\cdot\vert^{-}_{2-5i}$
0	0	0.0	0	0	0.0	0	0	0
1	1	0.1	1	1	0.1	1	1	1
2	2	0.2	2	2	0.2	2	2	2
$1+i$	18	0.3	$1+i$	13	0.3	3	$-2+2i$	$-2-2i$
$2+i$	19	0.4	$1+i$	14	0.4	4	$-1+2i$	$-1-2i$
$3+i$	20	0.5	$1+2i$	25	0.5	5	$2i$	$-2i$
$1+2i$	6	0.6	$2+2i$	26	0.6	6	$1+2i$	$1-2i$
$2+2i$	7	0.7	$1+3i$	8	0.7	7	$2+2i$	$2-2i$
i	17	1.1	i	12	1.1	8	$1-3i$	$1+3i$
$2i$	5	1.2	$2i$	24	1.2	9	$-3-i$	$-3+i$
$-1+i$	16	1.3	$-1+i$	11	1.3	10	$-2-i$	$-2+i$
$-1+2i$	4	1.4	$-1+2i$	23	1.4	11	$-1-i$	$-1+i$
$-1+3i$	21	1.5	$-2+i$	10	1.5	12	$-i$	i
$-2+i$	15	1.6	$-2+2i$	22	1.6	13	$1-i$	$1+i$
$-2+2i$	3	1.7	$-3+i$	9	1.7	14	$2-i$	$2+i$
-1	28	2.1	-1	28	2.1	15	$-2+i$	$-2-i$
-2	27	2.2	-2	27	2.2	16	$-1+i$	$-1-i$
$-1-i$	11	2.3	$-1-i$	16	2.3	17	i	$-i$
$-2-i$	10	2.4	$-2-i$	15	2.4	18	$1+i$	$1-i$
$-3-i$	9	2.5	$-1-2i$	4	2.5	19	$2+i$	$2-i$
$-1-2i$	23	2.6	$-2-2i$	3	2.6	20	$3+i$	$3-i$
$-2-2i$	22	2.7	$-1-3i$	21	2.7	21	$-1+3i$	$-1-3i$
$-i$	12	3.1	$-i$	17	3.1	22	$-2-2i$	$-2+2i$
$-2i$	24	3.2	$-2i$	5	3.2	23	$-1-2i$	$-1+2i$
$1-i$	13	3.3	$1-i$	18	3.3	24	$-2i$	$2i$
$1-2i$	25	3.4	$1-2i$	6	3.4	25	$1-2i$	$1+2i$
$1-3i$	8	3.5	$2-i$	19	3.5	26	$2-2i$	$2+2i$
$2-i$	14	3.6	$2-2i$	7	3.6	27	-2	-2
$2-2i$	26	3.7	$3-i$	20	3.7	28	-1	-1

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$\langle\cdot\vert^{-}_{1+6i} \leftrightarrow \vert\cdot\vert^{+}_{37}$		Кодир. (1)	$\langle\cdot\vert^{-}_{1-6i} \leftrightarrow \vert\cdot\vert^{+}_{37}$		Кодир. (1)	$\vert\cdot\vert^{+}_{37} \leftrightarrow \langle\cdot\vert^{-}_{1+6i} \leftrightarrow \langle\cdot\vert^{-}_{1-6i}$		
0	0	0.0	0	0	0.0	0	0	0
1	1	0.1	1	1	0.1	1	1	1
2	2	0.2	2	2	0.2	2	2	2
3	3	0.3	3	3	0.3	3	3	3
1+i	7	0.4	1+i	32	0.4	4	−2+i	−2−i
2+i	8	0.5	2+i	33	0.5	5	−1+i	−1−i
3+i	9	0.6	1+2i	26	0.6	6	i	−i
1+2i	13	0.7	2+2i	27	0.7	7	1+i	1−i
2+2i	14	0.8	1+3i	20	0.8	8	2+i	2−i
3+2i	15	0.9	2+3i	21	0.9	9	3+i	3−i
i	6	1.1	i	31	1.1	10	−2+2i	−2−2i
2i	12	1.2	2i	25	1.2	11	−1+2i	−1−2i
3i	18	1.3	3i	19	1.3	12	2i	−2i
−1+i	5	1.4	−1+i	30	1.4	13	1+2i	1−2i
−1+2i	11	1.5	−1+2i	24	1.5	14	2+2i	2−2i
−1+3i	17	1.6	−2+i	29	1.6	15	3+2i	3−2i
−2+i	4	1.7	−2+2i	23	1.7	16	−2+3i	−2−3i
−2+2i	10	1.8	−3+i	28	1.8	17	−1+3i	−1−3i
−2+3i	16	1.9	−3+2i	22	1.9	18	3i	−3i
−1	36	2.1	−1	36	2.1	19	−3i	3i
−2	35	2.2	−2	35	2.2	20	1−3i	1+3i
−3	34	2.3	−3	34	2.3	21	2−3i	2+3i
−1−i	30	2.4	−1−i	5	2.4	22	−3−2i	−3+2i
−2−i	29	2.5	−2−i	4	2.5	23	−2−2i	−2+2i
−3−i	28	2.6	−1−2i	11	2.6	24	−1−2i	−1+2i
−1−2i	24	2.7	−2−2i	10	2.7	25	−2i	2i
−2−2i	23	2.8	−1−3i	17	2.8	26	1−2i	1+2i
−3−2i	22	2.9	−2−3i	16	2.9	27	2−2i	2+2i
−i	31	3.1	−i	6	3.1	28	−3−i	−3+i
−2i	25	3.2	−2i	12	3.2	29	−2−i	−2+i
−3i	19	3.3	−3i	18	3.3	30	−1−i	−1+i
1−i	32	3.4	1−i	7	3.4	31	−i	i
1−2i	26	3.5	1−2i	13	3.5	32	1−i	1+i
1−3i	20	3.6	2−i	8	3.6	33	2−i	2+i
2−i	33	3.7	2−2i	14	3.7	34	−3	−3
2−2i	27	3.8	3−i	9	3.8	35	−2	−2
2−3i	21	3.9	3−2i	15	3.9	36	−1	−1

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$<\cdot\vert^{-}_{4+5i}\leftrightarrow\vert\cdot\vert^{+}_{41}$		(1) Кодир.	$\vert\cdot\vert^{+}_{41}$	(1) Кодир.	$\vert\cdot\vert_{41}\leftrightarrow<\cdot\vert^{-}_{4+5i}\leftrightarrow<\cdot\vert^{-}_{4-4i}$		
0	0	0.0	[illegible]	0.0	0	0	0
1	1	0.1	[illegible]	0.1	1	1	1
2	2	0.2	2	0.2	2	2	2
3	3	0.3	3	0.3	3	3	3
4	4	0.4	4	0.4	4	4	4
$1+i$	33	0.5	10	0.5	5	$4i$	$-4i$
$2+i$	34	0.6	11	0.6	6	$-3-i$	$-3+i$
$3+i$	35	0.7	12	0.7	7	$-2-i$	$-2+i$
$1+2i$	24	0.8	19	0.8	8	$-1-i$	$-1+i$
$2+2i$	25	0.9	20	0.9	9	$-i$	i
$1+3i$	15	0.10	28	0.10	10	$1-i$	$1+i$
i	32	1.1	9	1.1	11	$2-i$	$2+i$
$2i$	23	1.2	18	1.2	12	$3-i$	$3+i$
$3i$	14	1.3	27	1.3	13	$-1+3i$	$-1-3i$
$4i$	5	1.4	36	1.4	14	$3i$	$-3i$
$-1+i$	31	1.5	8	1.5	15	$1+3i$	$1-3i$
$-1+2i$	22	1.6	17	1.6	16	$-2-2i$	$-2+2i$
$-1+3i$	18	1.7	26	1.7	17	$-1-2i$	$-1+2i$
$-2+i$	30	1.8	7	1.8	18	$-2i$	$2i$
$-2+2i$	21	1.9	16	1.9	19	$1-2i$	$1+2i$
$-3+i$	29	1.10	6	1.10	20	$2-2i$	$2+2i$
-1	40	2.1	40	2.1	21	$-2+2i$	$-2-2i$
-2	39	2.2	39	2.2	22	$-1+2i$	$-1-2i$
-3	38	2.3	38	2.3	23	$2i$	$-2i$
-4	37	2.4	37	2.4	24	$1+2i$	$1-2i$
$-1-i$	8	2.5	31	2.5	25	$2+2i$	$2-2i$
$-2-i$	7	2.6	30	2.6	26	$-1-3i$	$-1+3i$
$-3-i$	6	2.7	29	2.7	27	$-3i$	$3i$
$-1-2i$	17	2.8	22	2.8	28	$1-3i$	$1+3i$
$-2-2i$	16	2.9	21	2.9	29	$-3+i$	$-3-i$
$-1-3i$	26	2.10	13	2.10	30	$-2+i$	$-2-i$
$-i$	9	3.1	32	3.1	31	$-1+i$	$-1-i$
$-2i$	18	3.2	23	3.2	32	i	$-i$
$-3i$	27	3.3	14	3.3	33	$1+i$	$1-i$
$-4i$	36	3.4	5	3.4	34	$2+i$	$2-i$
$1-i$	10	3.5	33	3.5	35	$3+i$	$3-i$
$1-2i$	19	3.6	24	3.6	36	$-4i$	$4i$
$1-3i$	28	3.7	15	3.7	37	-4	-4
$2-i$	11	3.8	34	3.8	38	-3	-3
$2-2i$	20	3.9	25	3.9	39	-2	-2
$3-i$	12	3.10	35	3.10	40	-1	-1

Key: (1). coding.

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Table of modular addition.

(1) no mod $2+3i$

+	0	1	2	9	8	3	7	12	11	4	5	10	6
9	0.0	0.1	0.2	0.3	1.1	1.2	1.3	2.1	2.2	2.3	3.1	3.2	3.3
1	0.1	0.2	1.2	3.2	0.3	2.3	1.1	0.0	2.1	3.1	3.3	2.2	1.3
2	0.2	1.2	2.3	2.2	3.2	3.1	0.3	0.1	0.0	3.3	1.3	2.1	1.1
9	0.3	3.2	2.2	3.1	2.3	2.1	1.2	1.1	1.3	0.0	0.1	3.3	0.2

(1) no mod $2-3i$

+	0	1	2	6	5	10	4	12	11	7	8	3	9
0	0.0	0.1	0.2	0.3	1.1	1.2	1.3	2.1	2.2	2.3	3.1	3.2	3.3
1	0.1	0.2	3.2	2.3	0.3	2.2	1.1	0.0	2.1	3.1	3.3	1.3	1.2
2	0.2	3.2	1.3	3.1	2.3	2.1	0.3	0.1	0.0	3.3	1.2	1.1	2.2
6	0.3	2.3	3.1	2.1	2.2	3.2	1.2	1.1	1.3	0.0	0.1	3.3	0.2

(1) no mod $1+4i$

+	0	1	2	5	6	4	8	8	7	16	15	12	11	13	9	14	19
0	0.0	0.1	0.2	0.3	0.4	1.1	1.2	1.3	1.4	2.1	2.2	2.3	2.4	3.1	3.2	3.3	3.4
1	0.1	0.2	1.3	0.4	1.4	0.3	3.2	1.1	1.2	0.0	2.1	3.1	2.3	3.3	3.4	2.2	2.4
2	0.2	1.3	1.1	1.4	1.2	0.4	3.4	0.3	3.2	0.1	0.0	3.3	3.1	2.2	2.4	2.1	2.3
9	0.3	0.4	1.4	3.4	2.4	3.2	3.1	1.2	2.3	1.1	1.3	0.0	2.1	0.1	3.3	0.2	2.2
6	0.4	1.4	1.2	2.4	2.3	3.4	3.3	3.2	3.1	0.3	1.1	0.1	0.0	0.2	2.2	1.3	2.1

(1) no mod $1-4i$

+	0	1	2	14	10	13	9	12	11	16	15	3	7	4	8	5	6
0	0.0	0.1	0.2	0.3	0.4	1.1	1.2	1.3	1.4	2.1	2.2	2.3	2.4	3.1	3.2	3.3	3.4
1	0.1	0.2	2.3	2.2	1.4	0.3	0.4	1.1	1.3	0.0	2.1	3.1	3.2	3.3	1.2	3.4	2.4
2	0.2	2.3	3.1	2.1	1.3	2.2	1.4	0.3	1.1	0.1	0.0	3.3	1.2	3.4	0.4	2.4	3.2
14	0.3	2.2	2.1	1.4	2.4	0.4	3.4	1.2	3.2	1.1	1.3	0.0	3.1	0.1	3.3	0.2	2.3
10	0.4	1.4	1.3	2.4	2.3	3.4	0.2	3.3	3.1	1.2	3.2	1.1	0.0	0.3	0.1	2.2	2.1

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(1) по mod 3+4i

+	0	1	2	3	19	20	12	18	11	4	17	10	16	24	23	22	6	5	13	7	14	21	8	15	9
0	0.0	0.1	0.2	0.3	0.4	0.5	0.6	1.1	1.2	1.3	1.4	1.5	1.6	2.1	2.2	2.3	2.4	2.5	2.6	3.1	3.2	0.3	3.4	3.5	3.6
1	0.1	0.2	0.3	1.3	0.5	3.3	2.6	0.4	0.6	2.5	1.1	1.2	1.4	0.0	2.1	2.2	3.1	2.4	3.2	3.4	3.5	2.3	3.6	1.6	1.5
2	0.2	0.3	1.3	2.5	3.3	2.3	3.2	0.5	2.6	2.4	0.4	0.6	1.1	0.1	0.0	2.1	3.4	3.1	3.5	3.6	1.6	2.2	1.5	1.4	1.2
3	0.3	1.3	2.5	2.4	2.3	2.2	3.5	3.3	3.2	3.1	0.5	2.6	0.4	0.2	0.1	0.0	3.6	3.4	1.6	1.5	1.4	2.1	1.2	1.1	0.6
19	0.4	0.5	3.3	2.3	2.6	3.2	2.4	0.6	2.5	2.2	1.2	1.3	1.5	1.1	1.4	1.6	0.0	2.1	3.1	0.1	3.4	3.5	0.2	0.6	0.3
20	0.5	3.3	2.3	2.2	3.2	3.5	3.1	2.6	2.4	2.1	0.6	2.5	1.2	0.4	1.1	1.4	0.1	0.0	3.4	0.2	3.6	1.6	0.3	1.5	1.3
12	0.6	2.6	3.2	3.5	2.4	3.1	2.1	2.5	2.2	1.6	1.3	2.3	0.3	1.2	1.5	3.6	1.1	1.4	0.0	0.4	0.1	3.4	0.5	0.2	3.3

(1) по mod 3—4i

+	0	1	2	3	8	9	15	7	14	21	6	13	5	24	23	22	17	16	10	18	11	4	19	12	20
0	0.0	0.1	0.2	0.3	0.4	0.5	0.6	1.1	1.2	1.3	1.4	1.5	1.6	2.1	2.2	2.3	2.4	2.5	2.6	3.1	3.2	3.3	3.4	3.5	3.6
1	0.1	0.2	0.3	3.3	0.5	2.6	2.5	0.4	0.6	2.3	1.1	1.2	1.4	0.0	2.1	2.2	3.1	2.4	3.2	3.4	3.5	1.6	3.6	1.5	1.3
2	0.2	0.3	3.3	1.6	2.6	3.2	2.4	0.5	2.5	2.2	0.4	0.6	1.1	0.1	0.0	2.1	3.4	3.1	3.5	3.6	1.5	1.4	1.3	1.2	2.3
3	0.3	3.3	1.6	1.4	3.2	3.5	3.1	2.6	2.4	2.1	0.5	2.5	0.4	0.2	0.1	0.0	3.6	3.4	1.5	1.3	1.2	1.1	2.3	0.6	2.2
8	0.4	0.5	2.6	3.2	2.5	2.4	2.2	0.6	2.3	3.3	1.2	1.3	1.5	1.1	1.4	1.6	0.0	2.1	3.1	0.1	3.4	3.5	0.2	2.2	0.3
9	0.5	2.6	3.2	3.5	2.4	3.1	2.1	2.5	2.2	1.6	0.6	2.3	1.2	0.4	1.1	1.4	0.1	0.0	3.4	0.2	3.6	1.5	0.3	1.3	3.3
15	0.6	2.5	2.4	3.1	2.2	2.1	1.6	2.3	3.3	3.2	1.3	0.3	3.6	1.2	1.5	3.5	1.1	1.4	0.0	0.4	0.1	3.4	0.5	0.2	2.6

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(1) no mod $2+5i$

+	0	1	2	18	19	20	6	7	17	5	16	4	21	15	3	28	27	11	10	9	23	22	12	24	13	25	8	14	26
0	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	1.1	1.2	1.3	1.4	1.5	1.6	1.7	2.1	2.2	2.3	2.4	2.5	2.6	2.7	3.1	3.2	3.3	3.4	3.5	3.6	3.7
1	0.1	0.2	1.7	0.4	0.5	1.5	0.7	3.5	0.3	0.6	1.1	1.2	2.7	1.3	1.4	0.0	2.1	3.1	2.3	2.4	3.2	2.6	3.3	3.4	3.6	3.7	2.5	1.6	2.2
2	0.2	1.7	1.4	0.5	1.5	2.7	3.5	2.5	0.4	0.7	0.3	0.6	2.6	1.1	1.2	0.1	0.0	3.3	3.1	2.3	3.4	3.2	3.6	3.7	1.6	2.2	2.4	1.3	2.1
18	0.3	0.4	0.5	0.7	3.5	2.5	3.2	3.4	0.6	2.6	1.2	2.7	2.4	1.4	1.5	1.1	1.3	0.0	2.1	2.2	3.1	2.3	0.1	3.3	0.2	3.6	3.7	1.7	1.6
19	0.4	0.5	1.5	3.5	2.5	2.4	3.4	3.7	0.7	3.2	0.6	2.6	2.3	1.2	2.7	0.3	1.1	0.1	0.0	2.1	3.3	3.1	0.2	3.6	1.7	1.6	2.2	1.4	1.3
20	0.5	1.5	2.7	2.5	2.4	2.3	3.7	2.2	3.5	3.4	0.7	3.2	3.1	0.6	2.6	0.4	0.3	0.2	0.1	0.0	3.6	3.3	1.7	1.6	1.4	1.3	2.1	1.2	1.1
6	0.6	0.7	3.5	3.2	3.4	3.7	3.1	3.3	2.6	2.3	2.7	2.4	2.2	1.5	2.5	1.2	1.4	1.1	1.3	1.6	0.0	2.1	0.3	0.1	0.4	0.2	3.6	0.5	1.7
7	0.7	3.5	2.5	3.4	3.7	2.2	3.3	3.6	3.2	3.1	2.6	2.3	2.1	2.7	2.4	0.6	1.2	0.3	1.1	1.3	0.1	0.0	0.4	0.2	0.5	1.7	1.6	1.5	1.4

(1) no mod $2-5i$

+	0	1	2	13	14	25	26	8	12	24	11	23	10	22	9	28	27	16	15	4	3	21	17	5	18	6	19	7	20
0	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	1.1	1.2	1.3	1.4	1.5	1.6	1.7	2.1	2.2	2.3	2.4	2.5	2.6	2.7	3.1	3.2	3.3	3.4	3.5	3.6	3.7
1	0.1	0.2	2.6	0.4	2.4	0.6	2.2	1.7	0.3	0.5	1.1	1.2	1.3	1.4	1.5	0.0	2.1	3.1	2.3	3.2	2.5	1.6	3.3	3.4	3.5	3.6	3.7	0.7	2.7
2	0.2	2.6	2.5	2.4	2.3	2.2	2.1	1.5	0.4	0.6	0.3	0.5	1.1	1.2	1.3	0.1	0.0	3.3	3.1	3.4	3.2	1.4	3.5	3.6	3.7	0.7	2.7	1.7	1.6
13	0.3	0.4	2.4	0.6	2.2	1.7	1.5	2.7	0.5	0.7	1.2	3.6	1.4	3.4	1.6	1.1	1.3	0.0	2.1	3.1	2.3	3.2	0.1	3.3	0.2	3.5	2.6	3.7	2.5
14	0.4	2.4	2.3	2.2	2.1	1.5	1.3	1.6	0.6	1.7	0.5	0.7	1.2	3.6	1.4	0.3	1.1	0.1	0.0	3.3	3.1	3.4	0.2	3.5	2.6	3.7	2.5	2.7	3.2
25	0.5	0.6	2.2	1.7	1.5	2.7	1.6	2.5	0.7	3.7	3.6	3.5	3.4	3.3	3.2	1.2	1.4	1.1	1.3	0.0	2.1	3.1	0.3	0.1	0.4	0.2	2.4	2.6	2.3
26	0.6	2.2	2.1	1.5	1.3	1.6	1.4	3.2	1.7	2.7	0.7	3.7	3.6	3.5	3.4	0.5	1.2	0.3	1.1	0.1	0.0	3.3	0.4	0.2	2.4	2.6	2.3	2.5	3.1
8	0.7	1.7	1.5	2.7	1.6	2.5	3.2	2.3	3.7	2.6	3.5	0.2	3.3	0.1	3.1	3.6	3.4	1.2	1.1	1.1	1.3	0.0	0.5	0.3	0.6	0.4	2.2	2.4	2.1

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(1) mo mod 1+6l

+	0	1	2	3	7	8	9	13	14	15	6	12	18	5	11	17	4	10	16
0	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9
1	0.1	0.2	0.3	1.7	0.5	0.6	1.8	0.8	0.9	1.9	0.4	0.7	3.3	1.1	1.2	1.3	1.4	1.5	1.6
2	0.2	0.3	1.7	1.4	0.6	1.8	1.5	0.9	1.9	1.6	0.5	0.8	3.6	0.4	0.7	3.3	1.1	1.2	1.3
3	0.3	1.7	1.4	1.1	1.8	1.5	1.2	0.9	1.6	1.3	0.6	0.9	3.9	0.5	0.8	3.6	0.4	0.7	3.3
7	0.4	0.5	0.6	1.8	0.8	0.9	1.9	3.6	3.9	2.9	0.7	3.3	3.2	1.2	1.3	2.7	1.5	1.6	2.8
8	0.5	0.6	1.8	1.5	0.9	1.9	1.6	3.9	2.9	2.8	0.8	3.6	3.5	0.7	3.3	3.2	1.2	1.3	2.7
9	0.6	1.8	1.5	1.2	1.9	1.6	1.3	2.9	2.8	2.7	0.9	3.9	3.8	0.8	3.6	3.5	0.7	3.3	3.2
13	0.7	0.8	0.9	0.9	3.6	3.9	2.9	3.5	3.8	2.6	3.3	3.2	3.1	1.3	2.7	2.4	1.6	2.8	2.5
14	0.8	0.9	1.9	1.6	3.9	2.9	2.8	3.8	2.6	2.5	3.6	3.5	3.4	3.3	3.2	3.1	1.3	2.7	2.4
15	0.9	1.9	1.6	1.3	2.9	2.8	2.7	2.6	2.5	2.4	3.9	3.8	3.7	3.6	3.5	3.4	3.3	3.2	3.1

+	36	35	34	30	29	28	24	23	22	31	25	19	32	26	20	33	27	21
0	2.1	2.2	2.3	2.4	2.5	2.6	2.7	2.8	2.9	3.1	3.2	3.3	3.4	3.5	3.6	3.7	3.8	3.9
1	0.0	2.1	2.2	3.1	2.4	2.5	3.2	2.7	2.8	3.4	3.5	3.6	3.7	3.8	3.9	2.3	2.6	2.9
2	0.1	0.0	2.1	3.4	3.1	2.4	3.5	3.2	2.7	3.7	3.8	3.9	2.3	2.6	2.9	2.2	2.5	2.8
3	0.2	0.1	0.0	3.7	3.4	3.1	3.8	3.5	3.2	2.3	2.6	2.9	2.2	2.5	2.8	2.1	2.4	2.7
7	1.1	1.4	1.7	0.0	2.1	2.2	3.1	2.4	2.5	0.1	3.4	3.5	0.6	3.7	3.8	0.3	2.3	2.6
8	0.4	1.1	1.4	0.1	0.0	2.1	3.4	3.1	2.4	0.2	3.7	3.8	0.3	2.3	2.6	1.7	2.2	2.5
9	0.5	0.4	1.1	0.2	0.1	0.0	3.7	3.4	3.1	0.3	2.3	2.6	1.7	2.2	2.5	1.4	2.1	2.4
13	1.2	1.5	1.8	1.1	1.4	1.7	0.0	2.1	2.2	0.4	0.1	3.4	0.5	0.2	3.7	0.6	0.3	2.3
14	0.7	1.2	1.5	0.4	1.1	1.4	0.1	0.0	2.1	0.5	0.2	3.7	0.6	0.3	2.3	1.8	1.7	2.2
15	0.8	0.7	1.2	0.5	0.4	1.1	0.2	0.1	0.0	0.6	0.3	2.3	1.8	1.7	2.2	1.5	1.4	2.1

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⊘ по mod 1—6i

+	0	1	2	3	32	33	26	27	20	21	31	25	19	30	24	29	23	28	22
0	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0,7	0.8	0.9	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1,9
1	0.1	0.2	0.3	2.5	0.5	2.3	0.7	1.8	0.9	1.9	0.4	0.6	0.8	1.1	1.2	1.4	1.5	1.6	1.7
2	0.2	0.3	2.5	2.4	2.3	2.2	1.8	1.6	1.9	1.7	0.5	0.7	0.9	0.4	0.6	1.1	1.2	1.4	1.5
3	0.3	2,5	2.4	3.2	2.2	2.1	1.6	1.4	1.7	1.5	2.3	1.8	1.9	0.5	0.7	0.4	0.6	1.1	1.2
32	0.4	0.5	2.3	2.2	0.7	1.8	0.9	1.9	3.9	2.9	0.6	0.8	3.7	1.2	1.3	1.5	3.3	1.7	2.8
33	0.5	2.3	2.2	2.1	1.8	1.6	1.9	1.7	2.9	2.8	1.8	0.9	3.9	0.6	0.8	1.2	1.3	1.5	3.3
26	0.6	0.7	1.8	1.6	0.9	1.9	3.9	2.9	3.8	2.7	0.8	3.7	3.6	1.3	3.5	3.3	3.2	2,8	2.6
27	0.7	1.8	1.6	1.4	1.9	1.7	2.9	2.8	2.7	2.6	0.8	3.9	3.8	0.8	3.7	1.3	3.5	3.3	3.2
20	0.8	0.9	1.9	1.7	3.9	2.9	3,8	2.7	0.3	2.5	3.7	3.6	0.2	3.5	3.4	3.2	3.1	2.6	2.4
21	0.9	1.9	1.7	1.5	2.9	2.8	2.7	2.6	2.5	2.4	3.9	3.8	0.3	3.7	3.6	3.5	3.4	3.2	3.1

+	36	35	34	5	4	11	10	17	16	6	12	18	7	13	8	14	9	15
0	2.1	2.2	2.3	2.4	2.5	2.6	2.7	2.8	2.9	3.1	3.2	3.3	3.4	3.5	3.6	3.7	3.8	3,9
1	0.0	2.1	2.2	3.1	2.4	3.2	2.6	3.3	2.8	3.4	3.5	1.3	3.6	3.7	3.8	3.9	2.7	2.9
2	0.1	0.0	2.1	3.4	3.1	3.5	3.2	1.3	3.3	3.6	3.7	0.8	3.8	3.9	2.7	2.9	2.6	2.8
3	0.2	0.1	0,0	3.6	3.4	3.7	3.5	0.8	1.3	3.8	3.9	0.9	2.7	2.9	2.6	2.8	3.2	3.3
32	1.1	1.4	1.6	0.0	2.1	3.1	2.4	3.2	2.6	0.1	3.4	3.5	0.2	3.6	0.3	3.8	2.5	2.7
33	0.4	1.1	1.4	0.1	0.0	3.4	3.1	3.5	3.2	0.2	3.6	3.7	0.3	3.8	2.5	2.7	2.4	2.6
26	1.2	1.5	1.7	1.1	1.4	0.0	2.1	3.1	2.4	0.4	0.1	3.4	0.5	0.2	2.3	0.3	2.2	2.5
27	0.6	1.2	1.5	0.4	1.1	0.1	0.0	3.4	3.1	0.5	0.2	3.6	2.3	0.3	2.2	2.5	2.1	2.4
20	1.3	3.3	2.8	1.2	1.5	1.1	1.4	0.0	2.1	0.6	0.4	0.1	0.7	0.5	1.8	2.3	1.6	2.2
21	0.8	1,3	3,3	0.6	1.2	0.4	1,1	0.10	0.0	0.7	0.5	0.2	1.8	2.3	1.6	2.2	1.4	2.1

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(1) no mod 4+5i

+	0	1	2	3	4	33	34	35	24	25	15	32	23	14	5	31	22	13	30	21
0	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	0.10	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9
1	0.1	0.2	0.3	0.4	1.4	0.6	0.7	3.4	0.9	2,10	2.9	0.5	0.8	0.10	2.7	1.1	1.2	1.3	1.5	1.6
2	0.2	0.3	0.4	1.4	2.7	0.7	3.4	2.4	2.10	3.3	2.8	0.6	0.9	2.9	2.6	0.5	0.8	0.10	1.1	1.2
3	0.3	0.4	1.4	2.7	2.6	3.4	2.4	2.3	3.3	3.7	3.2	0.7	2.10	2.8	2.5	0.6	0.9	2.9	0.5	0.8
4	0.4	1.4	2.7	2.6	2.5	2.4	2.3	2.2	3.7	1.10	3.6	3.4	3.3	3.2	3.1	0.7	2.10	2.8	0.6	0.9
33	0.5	0.6	0.7	3.4	2.4	0.9	2.10	3.3	2.9	2.8	2.6	0.8	0.10	2.7	2.3	1.2	1.3	1.4	1.6	1.7
34	0.6	0.7	3.4	2.4	2.3	2.10	3.3	3.7	2.8	3.2	2.5	0.9	3.9	2.6	2.2	0.8	0.10	2.7	1.2	1.3
35	0.7	3.4	2.4	2.3	2.2	3.3	3.7	1.10	3.2	3.6	3.1	2.10	3.8	2.5	2.1	0.9	2.9	2.6	0.8	0.10
24	0.8	0.9	2,10	3.3	3,7	3.7	2,8	3.2	2.6	2.5	2.2	0.10	2.7	2.3	1.10	1.3	1.4	2.4	1.7	3.5
25	0.9	2.10	3.3	3.7	1.10	2.9	3.2	3.6	2.5	3.1	2.1	2.9	2.6	2.2	1.8	0.10	2.7	2.3	1.3	3.8
15	0.10	2.9	2.8	3.2	3.6	2.8	2.5	3.1	2.2	2.1	1.8	2.7	2.3	1.10	3.6	1.4	2.4	3.7	0.4	3.4

+	29	40	39	38	37	8	7	6	17	16	26	9	18	27	36	10	19	28	11	20	12
0	1.10	2.1	2.2	2.3	2.4	2.5	2.6	2.7	2.8	2.9	2.10	3.1	3.2	3.3	3.4	3.5	3.6	3.7	3.8	3.9	3.10
1	1.8	0.0	2.1	2.2	2.3	3.1	2.5	2.6	3.2	2.8	3.3	3.5	3.6	3.7	2.4	3.8	3.9	1.10	3.10	1.9	1.7
2	1.5	0.1	0.0	2.1	2.2	3.5	3.1	2.5	3.6	3.2	3.7	3.8	3.9	1.10	2.3	3.10	1.9	1.8	1.7	1.6	1.3
3	1.1	0.2	0.1	0.0	2.1	3.8	3.5	3.1	3.9	3.6	1.10	3.10	1.9	1.8	2.2	1.7	1.6	1.5	1,3	1.2	0.10
4	0.5	0.3	0.2	0.1	0.0	3.10	3.8	3.5	1.9	3.9	1.8	1.7	1.6	1.5	2.1	1.3	1.2	1.1	0,10	0.8	2.9
33	1.9	1.1	1.5	1.8	1.10	0.0	2.1	2.2	3.1	2.5	3.2	0.1	3.5	3.6	3.7	0.2	3.8	3.9	0.3	3.10	0.4
34	1.6	0.5	1.1	1.5	1.8	0.1	0.0	2.1	3.5	3.1	3.6	0.2	3.8	3.9	1.10	0.3	3.10	1.9	0.4	1.7	1.4
35	1.2	0.6	0.5	1.1	1.5	0.2	0.1	0.0	3.8	3.5	3.9	0.3	3.10	1.9	1.8	0.4	1.7	1.6	1.4	1.3	2.7
24	3.2	2.2	1.6	1.9	3.9	1.1	1.5	1.8	0.0	2.1	3.1	0.5	0.1	3.5	3.6	0.6	0.2	3.8	0.7	0.3	3.4
25	3.6	0.8	2.2	1.6	1.6	0.5	1.1	1.5	0.1	0.0	3.5	0.6	0.2	3.8	3.9	0.7	0.3	3.10	3.4	0.4	0.7
15	0.3	1.3	1.7	3.10	3.8	1.2	1.6	1.9	1.6	1.5	0.0	0.8	0.5	0.1	3.5	0.9	0,6	0.2	2.10	0.7	3.3

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(1) по mod 4 -5i

+	0	1	2	3	4	10	11	12	19	20	28	9	18	27	36	8	17	26	7	16
0	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	0.10	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9
1	0.1	0.2	0.3	0.4	3.4	0.6	0.7	2.10	0.9	2.9	2.7	0.5	0.8	0.10	2.4	1.1	1.2	1.3	1.5	1.6
2	0.2	0.3	0.4	3.4	1.10	0.7	2.10	2.3	2.9	2.8	2.6	0.6	0.9	2.7	2.3	0.5	0.8	0.10	1.1	1.2
3	0.3	0.4	3.4	1.10	1.8	2.10	3.3	3.7	2.2	3.2	2.5	0.7	2.9	2.6	2.2	0.6	0.9	2.7	0.5	0.8
4	0.4	3,4	1.10	1.8	1.5	3.3	3.7	1.9	3.2	3.6	3.1	2.10	2.8	2.5	2.1	0.7	2.9	2.6	0.6	0.9
10	0.5	0.6	0.7	2.10	3.3	0.9	2.9	2.8	2.7	2.6	2.3	0.8	0.10	2.4	3.4	1.2	1.3	1.4	1.6	1.7
11	0.6	0.7	2.10	3.3	3.7	2.9	2.8	3.2	2.6	2.5	2.2	0.9	2.7	2.3	1.10	0.8	0.10	2.4	1.2	1.30
12	0.7	2.10	3.3	3.7	1.9	2.8	3.2	3.6	2.5	3.1	2.1	2.9	2.6	2.2	1.8	0.9	2.7	2.3	0.8	0.10
19	0.8	0.9	2.9	2.2	3.2	2.7	2.6	2.5	2.3	2.2	1.10	0.10	2.4	3.4	3.3	1.3	1.4	0.4	1.7	3.10
20	0.9	2.9	2.8	3.2	3.6	2.6	2.5	3.1	2.2	2.1	1.8	2.7	2.3	1.10	3.7	0.10	2.4	3.4	1.3	1.4
28	0.10	2.7	2.6	2.5	3.1	2.8	2.2	2.1	1.10	1.8	3.7	2.4	3,4	3.3	3.2	1.4	0.4	2.10	3.10	0.3

+	6	40	39	38	37	31	30	29	22	21	13	32	23	14	5	33	24	15	34	25	35
0	1.10	2.1	2.2	2.3	2.4	2.5	2.6	2.7	2,8	2.9	2.10	3.1	3.2	3.3	3.4	3.5	3.6	3.7	3.8	3.9	3.10
1	1.8	0.0	2.1	2.2	2.3	3.1	2.5	2.6	3.2	2.8	3.3	3.5	3.6	2.7	1.10	3.8	3.9	1.9	3.10	1.7	1.4
2	1.5	0.1	0.0	2.1	2.2	3.5	3.1	2.5	3.6	3.2	3.7	3.8	3.9	1.9	1.8	3.10	1.7	1.6	1.4	1.3	2.4
3	1.1	0.2	0.1	0.0	2.1	3.8	3.5	3.1	3.9	3.6	1.9	3.10	1.7	1.6	1.5	1.4	1.3	1.2	2.4	0.10	2.3
4	0.5	0.3	0.2	0.1	0.0	3.10	3.8	3.5	1.7	3.9	1.6	1.4	1.3	1.2	1.1	2.4	0.10	0.8	2.3	2.7	2.2
10	1.9	1.1	1.5	1.8	1.10	0.0	2.1	2.2	3.1	2.5	3.2	0.1	3.5	3.6	3.7	0.2	3.8	3.9	0.3	3.10	0.4
11	1.6	0.5	1.1	1.5	1.8	0.1	0.0	2.1	3.5	3.1	3.6	0.2	3.8	3.9	1.9	0.3	3.10	1.7	0.4	1.4	3.4
12	1.2	0.6	0.5	1.1	1.5	0.2	0.1	0.0	3.8	3.5	3,9	0.3	3.10	1.7	1.6	0.4	1.4	1.3	3.4	2.4	1.10
19	3.9	1.2	1.6	1.9	3.7	1.1	1.5	1.8	0.0	2.1	3.1	0.5	0.1	3.5	3.6	0.6	0.2	3.8	0.7	0.3	2.10
20	1.7	0.8	1.2	1.6	1.9	0.5	1.1	1.5	0.1	0.0	3.5	0.6	0.2	3.8	3.9	0.7	0.3	3.10	3.10	0.4	3.3
28	3.8	1.3	1.7	3.9	3.6	1.2	1.6	3.7	1.1	1.5	0,0	0,8	0.5	0.1	3,5	0.9	0.6	0.2	2.9	0.7	2.8

Key: (1). ~~on~~ modulas.

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Table of modular multiplication.

(1) no mod $2+3i$

×	0	1	2	9
0	0.0			
1		0.1	0.2	0.3
2		0.2	2.3	3.1
9		0.3	3.1	1.2

(1) no mod $2-3i$

×	0	1	2	6
0	0.0			
1		0.1	0.2	0.3
2		0.2	1.3	2.1
6		0.3	2.1	1.2

(1) no mod $1+4i$

×	0	1	2	5	6
0	0.0				
1		0.1	0.2	0.3	0.4
2		0.2	1.1	3.4	2.3
5		0.3	3.4	1.2	3.1
6		0.4	2.3	3.1	0.2

(1) no mod $3-4i$

×	0	1	2	14	10
0	0.0				
1		0.1	0.2	0.3	0.4
2		0.2	3.1	1.4	2.3
14		0.3	1.4	1.2	3.1
10		0.4	2.3	3.1	2.2

(1) no mod $3+4i$

×	0	1	2	8	19	20	12
0	0.0						
1		0.1	0.2	0.3	0.4	0.5	0.6
2		0.2	1.3	2.4	2.6	8.5	2.1
8		0.3	2.4	3.6	3.1	1.5	1.2
19		0.4	2.6	3.1	1.2	2.5	0.3
20		0.5	3.5	1.5	2.5	0.0	3.5
12		0.6	2.1	1.2	0.3	8.5	0.4

(1) no mod $3-4i$

×	0	1	2	3	8	9	15
0	0.0						
1		0.1	0.2	0.3	0.4	0.5	0.6
2		0.2	3.8	1.4	2.5	3.1	1.6
3		0.3	1.4	0.5	2.1	0.2	3.6
8		0.4	2.5	2.1	1.2	2.3	3.6
9		0.5	3.1	0.2	2.3	1.4	2.6
15		0.6	1.6	3.6	3.6	2.6	0.0

(1) no mod $2+5i$

×	0	1	2	18	19	20	6	7
0	0.0							
1		0.1	0.2	0.3	0.4	0.5	0.6	0.7
2		0.2	1.4	0.7	2.5	2.3	3.1	3.6
18		0.3	0.7	1.2	2.6	3.1	1.5	2.4
19		0.4	2.5	2.6	3.3	1.7	3.2	3.6
20		0.5	2.3	3.1	1.7	2.6	1.4	3.2
6		0.6	3.1	1.5	3.2	1.4	0.7	3.3
7		0.7	3.6	2.4	3.6	3.2	3.3	0.5

(1) no mod $2-5i$

×	0	1	2	13	14	25	26	8
0	0.0							
1		0.1	0.2	0.3	0.4	0.5	0.6	0.7
2		0.2	2.5	0.6	2.1	2.7	0.3	2.3
13		0.3	0.6	1.2	0.7	3.4	3.5	3.1
14		0.4	2.1	0.7	1.6	0.2	2.3	0.5
25		0.5	2.7	3.4	0.2	2.3	1.1	0.6
26		0.6	0.3	3.5	2.3	1.1	1.5	3.2
8		0.7	2.3	3.1	0.5	0.6	3.2	3.4

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(1) no mod 1+6i

×	0	1	2	3	7	8	9	13	14	15
0	0.0									
1		0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
2		0.2	1.7	1.1	0.8	1.9	1.3	3.5	2.6	2.4
3		0.3	1.1	0.6	3.9	2.7	3.8	0.2	1.4	0.5
7		0.4	0.8	3.9	1.2	3.3	3.5	1.6	2.7	3.1
8		0.5	1.9	2.7	3.3	3.8	2.2	2.4	0.1	0.6
9		0.6	1.3	3.8	3.5	2.2	0.4	1.1	0.9	2.7
13		0.7	3.5	0.2	1.6	2.4	1.1	3.9	2.3	1.8
14		0.8	2.6	1.4	2.7	0.1	0.9	2.3	1.5	3.2
15		0.9	2.4	0.5	3.1	0.6	2.7	1.8	3.2	0.3

(1) no mod 4—6i

×	0	1	2	3	32	33	26	27	20	21
0	0.0									
1		0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
2		0.2	2.5	3.1	0.7	1.6	3.9	2.8	0.3	2.4
3		0.3	3.1	3.8	1.9	1.2	2.5	3.4	1.7	0.6
32		0.4	0.7	1.9	1.2	0.8	3.3	3.5	2.6	3.1
33		0.5	1.6	1.2	0.8	2.9	3.4	0.3	1.1	0.7
26		0.6	3.9	2.5	3.3	3.4	2.7	2.1	0.2	1.8
27		0.7	2.8	3.4	3.5	0.3	2.1	0.6	1.9	3.2
20		0.8	0.3	1.7	2.6	1.1	0.2	1.9	1.4	3.5
21		0.9	2.4	0.6	3.1	0.7	1.8	3.2	3.5	2.3

(1) no mod 4+5i

×	0	1	2	3	4	33	34	35	24	25	15
0	0.0										
1		0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	0.10
2		0.2	0.4	2.7	2.5	0.9	3.3	1.10	2.63	3.1	1.8
3		0.3	2.7	3.1	3.10	2.8	3.9	1.2	1.5	0.6	0.4
4		0.4	2.5	3.10	2.9	3.1	1.7	2.8	1.8	3.2	3.6
33		0.5	0.9	2.8	3.1	1.2	0.10	2.6	1.7	1.4	0.8
34		0.6	3.3	3.9	1.7	0.10	2.5	0.1	2.4	1.8	3.2
35		0.7	1.10	1.2	2.8	2.6	0.1	3.4	3.9	1.3	0.5
24		0.8	2.6	1.5	1.3	1.7	2.4	3.9	0.2	2.10	1.1
25		0.9	3.1	0.6	3.2	1.4	1.8	1.3	2.10	3.5	1.10
15		0.10	1.8	0.4	3.6	0.3	3.2	0.5	1.1	1.10	3.9

(1) no mod 4—5i

×	0	1	2	3	4	10	11	12	19	20	28
0	0.0										
1		0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	0.10
2		0.2	0.4	1.10	1.5	0.9	2.8	3.6	2.3	2.1	3.7
3		0.3	1,10	1.1	0.7	2.6	3.5	1.4	1.9	0.8	0.2
4		0.4	1.5	0.7	1.9	2.1	0.3	1.8	3.10	2.2	2.6
10		0.5	0.9	2.6	2.1	1.2	0.10	2.3	1.7	1.4	3.8
11		0.6	2.8	3.5	0.3	0.10	2.2	1.1	0.4	3.7	2.9
12		0.7	3.6	1.4	1.8	2.3	1.1	2.9	3.2	3.10	1.5
19		0.8	2.3	1.9	3.10	1.7	0.4	3.2	3.6	0.6	2.1
20		0.9	2.1	0.8	2.2	1.4	3.7	3.10	0.6	2.5	1.3
28		0.10	3.7	0.2	2.6	3.8	2.9	1.5	2.1	1.3	3.4

Key: (1). on.

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§6. Algorithm of conversion of decimal representation of ts.k.ch. into the nonpositional.

On the given one ts.k.ch. $z=A+Bi\in(Q_n)$ it is necessary to determine remainders/residues on pair-wise conjugated/combined bases/bases $p_k, \bar{p}_k$ ($1\leqslant k\leqslant n$) and to present these remainders/residues in accordance with the code accepted.

This problem solves the algorithm whose diagram is described below:

$$1)\ \langle A+Bi\,|_{\bar{q}_k} = |A|_{\bar{q}_k} + i\,|B|_{\bar{q}_k},$$
$$2)\ \| A|_{\bar{q}_k} + \rho_k |B|_{\bar{q}_k}|^{+}_{q_k} \Longleftrightarrow \langle z|_{\bar{p}_k} \longleftrightarrow (\text{код}),$$
$$3)\ \| A|_{\bar{q}_k} + \bar{\rho}_k |B|_{\bar{q}_k}|^{+}_{q_k} \Longleftrightarrow \langle z|_{\bar{\bar{p}}_k} \longleftrightarrow (\text{код}),$$

Key: (1). the code.

where ρ_k and $\bar{\rho}_k$ — coefficients of isomorphism in moduli/modules p_k and $\bar{p}_k$.

In other words, during the first stage is revealed/detected canonical deduction ts.k.ch. z=A+Bi on pair-wise conjugated/combined moduli/modules $p_k, \bar{p}_k$; the second and third stages are implemented in parallel, moreover by arithmetic conversion are subject real

deductions on the real moduli/modules.

Their practical realization is reduced to one recoding $(\rho_k \cdot |B|^{-}_{q_k})$ and modular addition, in this case it is important to note that in accordance with the principle of isomorphism the corresponding operations can be realized on the equipment, intended for the modular addition of composite deductions.

Example. Let us determine remainders/residues ts.k.ch. 15+21i on pair-wise conjugated/combined bases/bases $p_1=2+3i$, $\overline{p}_1=2-3i$, $p_2=1+4i$, $\overline{p}_2=1-4i$, $p_3=3+4i$, $\overline{p}_3=3-4i$ according to the algorithm outlined above:

$$1)\ <15+21i\,|^{-}_{13}=2-5i,$$
$$<15+21i\,|^{-}_{17}=-2+4i,$$
$$<15+21i\,|^{-}_{25}=-10-4i;$$
$$2)\ |2-\rho_1 5|^{+}_{13}=|2-8\cdot 5|^{+}_{13}<==><1\,|^{-}_{2+3i}\longleftrightarrow 0.1,$$
$$|-2+\rho_2 4|^{+}_{17}=|-2+4\cdot 4|^{+}_{17}<==><1-i\,|^{-}_{1+4i}\longleftrightarrow 3.3,$$
$$|-10-\rho_3 4|^{+}_{25}=|-10-18\cdot 4|^{+}_{25}<==><i\,|^{-}_{3+4i}\longleftrightarrow 1.1;$$
$$3)\ |2-\overline{\rho}_1 5|^{-}_{13}=|2-5\cdot 6|^{+}_{13}<==><-2i\,|^{-}_{2-3i}\longleftrightarrow 3.2,$$
$$|-2+\overline{\rho}_2 4|^{-}_{17}=|-2-4\cdot 4|^{+}_{17}<==><-1\,|^{-}_{1-4i}\longleftrightarrow 2.1,$$
$$|-10-\overline{\rho}_3 4|^{-}_{25}=|-10-8\cdot 4|^{+}_{25}<==><1-2i\,|^{-}_{3-4i}\longleftrightarrow 3.6.$$

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Thus, in the adapted system 15+21i-(0.1 ~~accepted~~, 3.2; 3.3, 2.1; 1.1, 3.5). For convenience in the calculations are given the tables (see

appendix 1), which realize the translation/conversion of canonical deductions into the deductions on moduli/modules $p_k, \bar{p}_k$, registered in accordance with the code accepted for the composite deductions.

§7. Translation algorithm of ts.k.ch. preset by the nonpositional code, into the mixed positional KOD.

Let us assume at first that ts.k.ch. $z \in (Q_n)$ is represented by the canonical nonpositional code

$$z \div (\overset{q_1}{w_1}, \overset{q_2}{w_2}, \ldots, \overset{q_n}{w_n}). \qquad (7.1)$$

It is necessary to determine components ζ_k of the mixed positional representation of number z:

$$z = \zeta_1 + \zeta_2 q_1 + \zeta_3 q_1 q_2 + \ldots + \zeta_n q_1 q_2 \ldots q_{n-1}, \qquad (7.2)$$

where

$$\zeta_k \in <\cdot\mid^{\pm}_{q_k}.$$

The unknown components can be defined as the remainders/residues of the following process of consecutive indexing (see the proof of theorem 3.3.2): z is divided into q_1: the obtained partial quotient $\omega^{(1)}$ is divided into q_2: partial quotient $\omega^{(2)}$ from the previous division is divided into q_3 and so forth.

This process is described by the following system of equalities:

$$
\begin{aligned}
&z = \zeta_1 + \omega^{(1)} q_1 && \zeta_1 = <z \mid^{\pm}_{q_1} \\
&\omega^{(1)} = \zeta_2 + \omega^{(2)} q_2 && \zeta_2 = <\omega^{(1)} \mid^{\pm}_{q_2} \\
&\omega^{(2)} = \zeta_3 + \omega^{(3)} q_3 && \zeta_3 = <\omega^{(2)} \mid^{\pm}_{q_3} \\
&\cdots\cdots\cdots && \cdots\cdots \\
&\omega^{(m-1)} = \zeta_m + \omega^{(m)} q_m && \zeta_m = <\omega^{m-1} \mid^{\pm}_{q_m} \\
&\cdots\cdots\cdots && \cdots\cdots \\
&\omega^{(n-1)} = \zeta_n && \zeta_n = <\omega^{n-1} \mid^{\pm}_{q_n}.
\end{aligned}
\tag{7.3}
$$

In view of the determination of range (Q_n), for any ts.k.ch. $z \in (Q_n)$ the process of calculations is completed, at least, at the n step/pitch.

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Let us paraphrase the realization of the described algorithm into the language of the nonpositional representation of numbers. For this purpose in equalities (7.3) let us pass to the deductions on moduli/modules q_r, then we obtain n of the systems of the comparisons:

$$
\left.
\begin{aligned}
&w_r \equiv \zeta_{1r} + \omega_r^{(1)} q_{1r} \\
&\omega_r^{(1)} \equiv \zeta_{2r} + \omega_r^{(2)} q_{2r} \\
&\cdots\cdots\cdots \\
&\omega_r^{(m-1)} \equiv \zeta_{mr} + \omega_r^{(m)} q_{mr} \\
&\cdots\cdots\cdots \\
&\omega_r^{(n-1)} \equiv \zeta_{nr}
\end{aligned}
\right\}
\begin{array}{l}
(\mathrm{mod}\ q_r) \\
1 \leqslant r \leqslant n,
\end{array}
\tag{7.4}
$$

where

$$
\begin{aligned}
&\zeta_{kr} = <\zeta_k \mid^{\pm}_{q_r}, \\
&\omega_r^{(m)} = <\omega^{(m)} \mid^{\pm}_{q_r}, \\
&q_{mr} = <q_m \mid^{+}_{q_r}.
\end{aligned}
$$

In accordance with recurrence formulae (7.4) the process of determining the numbers ζ_1, ζ_2, ... by consecutive indexing can be represented in the language of remainders/residues in the form of the following diagram:

the zero step/pitch:

$$\zeta_1 = w_1;$$

the first step/pitch:

$$\omega_r^{(1)} = \langle q_{1r}^{-1}\left(w_r - \langle w_1 \,|^{\pm}_{q_r}\right) |^{\pm}_{q_r}, \quad 2 \leqslant r \leqslant n,$$

moreover

$$\zeta_2 = \omega_2^{(1)} \in \langle \cdot \,|^{\pm}_{q_r};$$

the second step/pitch:

$$\omega_r^{(2)} = \langle q_{2r}^{-1}\left(\omega_r^{(1)} - \langle \omega_2^{(1)} \,|\right)^{\pm}_{q_r} |^{\pm}_{q_r}, \quad 3 \leqslant r \leqslant n,$$

moreover

$$\zeta_3 = \omega_3^{(2)} \in \langle \cdot \,|^{\pm}_{q_3};$$

the third step/pitch:

$$\omega_r^{(3)} = \langle q_{3r}^{-1}\left(\omega_r^{(2)} - \langle \omega_3^{(2)} \,|^{\pm}_{q_r}\right) |^{\pm}_{q_r}; \quad 4 \leqslant r \leqslant n,$$

$$\dots\dots\dots\dots\dots\dots\dots\dots$$

(m-1)-th step/pitch:

$$\omega_r^{(m-1)} = \langle q_{m-1,r}^{-1}\left(\omega_r^{(m-2)} - \langle \omega_{m-1}^{(m-2)} \,|^{\pm}_{q_r}\right) |^{\pm}_{q_r}, \quad m \leqslant r \leqslant n,$$

moreover

$$\zeta_m = \omega_m^{(m-1)} \theta < \cdot \mid_{q_m}^{\pm};$$

$$\cdots\cdots\cdots\cdots$$

(n-1) -th step/pitch:

$$\zeta_n = \omega_n^{(n-1)} = < q_{n-1,n}^{-1} (\omega_n^{(n-2)} - \omega_{n-1}^{(n-2)} \mid_{q_n}^{\pm}) \mid_{q_n}^{\pm}.$$

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A block-circuit recording of this algorithm is shown in figure 44.

In terms of its structure this diagram is similar to the diagram of the calculation of the divided differences in the theory of interpolation. In connection with this let us name its Δ- algorithm. a Δ- algorithm realizes the translation/conversion ts.k.ch., preset by the canonical nonpositional code, into the mixed positional code. In actuality each canonical deduction w_k is represented by the pair of deductions α_k, β_k on moduli/modules $p_k, \overline{p}_k$. With exception of recoding of the type

$$< - < x \mid_{q_k}^{\pm} \mid_{q_m}^{\pm} \qquad (7.5)$$

all stages of calculations on a Δ- algorithm modular, and therefore they can be realized in accordance with the rules of modular arithmetic in parallel and independently directly above the nonpositional representation of canonical deductions.

Recoding (7.5) requires the restoration/reduction of the value of canonical deduction on $\bmod q_n$. In connection with this let us pose the following problem.

Let w=x+iy - canonical (nonstandardized) deduction ts.k.ch. $z=A+Bi\in(Q_n)$ on mod q ($q=p\cdot\bar{p}$), i.e.,

$$x+iy=<A+Bi|_q^-.$$

It is known that

$$<x+iy|_p^-=\alpha,\ <x+iy|_{\bar{p}}^{=}=\beta. \qquad (7.6)$$

It is necessary to restore/reduce value x+iy.

First method. It is assumed that modulus/module p=c+di satisfies condition (c, d)=1.

ρ and $\bar{\rho}$ - coefficients of isomorphism respectively in moduli/modules p and $\bar{p}$.

Let us compare to deductions α, β the isomorphic deductions r_1 and r_2 on mod q.

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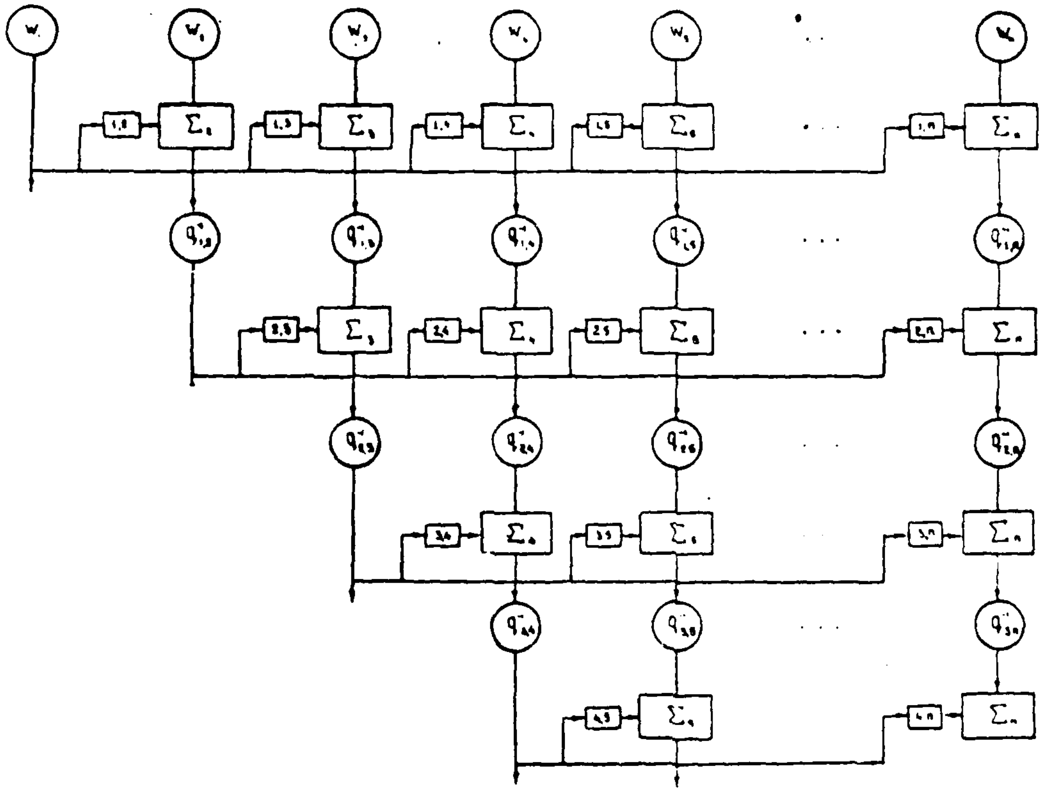

Fig. 44. Block diagram of translation algorithm ts.k.ch., preset by nonpositional code, into mixed positional code. $\boxed{\Sigma_k}$ — adder, which realizes operation $< x + ly|^{\pm}_{q_k}$.

$\boxed{k,m}$ — block of recoding, which realizes the operation: $\langle\langle -x|^{\pm}_{q_n}|^{\pm}_{q_m}$.

$q^{-1}_{k,r}$ — block of recoding, which realizes the operation: $< q^{-1}_{k,r}x|^{\pm}_{q_r}$.

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Then relationships/ratios (7.6) can be presented in the form

$$x+\bar{\rho}y\equiv r_2(\mathrm{mod}\, q).$$

From the latter/last system we obtain:

$$(\rho-\bar{\rho})\,y\equiv r_1-r_2\,(\mathrm{mod}\, q).$$

Therefore, when $(\rho-\bar{\rho},\ q)=1$, we have

$$y=\langle (r_1-r_2)\ (\rho-\bar{\rho})^{-1}\mid_{\bar{q}}$$

and

$$x=\langle r_1-\rho\langle (r_1-r_2)\ (\rho-\bar{\rho})^{-1}\mid_{\bar{q}}\mid_{\bar{q}}.$$

Second method. According to theorem 3.3.4 for $w=x+iy\in\langle\cdot\mid_{\bar{q}}$ we have

$$w=\xi+\eta p+\varepsilon q, \qquad (7.7)$$

where

$$\|\varepsilon\|=\begin{cases}0\\1\end{cases}.$$

Consequently, task is reduced to the determination of values ξ, η, ε. Since $\varepsilon\in\langle\cdot\mid_{\overline{1+2i}}$, the equality (7.7) can be considered as representation ts.k.ch. w in the mixed positional code with bases/bases p, $\bar{p}$, 1+2i. In that case for determining the parameter ε is necessary surplus information about value w. Specifically, it is necessary to know the deduction

$$\gamma=\langle w\mid_{\overline{1+2i}}.$$

Actually/really, it is obvious, that $\varepsilon=\alpha$. Since

$$\frac{w-\alpha}{p}=\eta+\varepsilon\bar{p},$$

that

$$\eta=<p^{-1}(\beta-<\alpha\mid_{\bar{p}})\mid_{\bar{p}}.$$

Further, from the fact that

$$\varepsilon=\frac{1}{\bar{p}}\left(\frac{w-\alpha}{p}-\eta\right),$$

follows

$$\varepsilon=<\bar{p}^{-1}(p^{-1}(\gamma-\alpha)-\eta)\mid_{1+2i}.$$

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Both methods it is not possible to consider sufficiently efficient, since the first method requires, at least, two recodings even one operations of modular addition, and the second - presence of surplus basis/base on each pair of the complex conjugate bases/bases is made, at least, for two strokes/cycles of modular addition.

Third method (tabular version). Since the pair of remainders/residues (α, β) on moduli/modules p, $\bar{p}$ of canonical deduction $w \subset <\cdot\mid_{\|p\|}$ uniquely determines value w, then value w can be assigned by table with two input: α, β.

Let us consider a question about the reduction of these tables in light of coding of deductions accepted.

We have

$$w = \alpha\overline{B} + \beta B + r \parallel p \parallel, \qquad (7.8)$$

where B=mp and $m = < p^{-1} |_{\overline{p}}$.

Let

$$\alpha = (k_1)\, n_1 = i^{k_1} n_1,$$
$$\beta = (k_2)\, n_2 = i^{k_2} n_2,$$

where n_1, n_2 - mantissa, k_1, k_2 - orders of deductions. Then relationship/ratio (7.8) can be presented in the form

$$w = i^{k_1}(n_1 \overline{B} + i^{|k_2-k_1|_4^+} n_2 B + r i^{|4-k_1|_4^+} \parallel p \parallel).$$

In the force of the property of the symmetry of deductions ts.k.ch., the prisoner into the parentheses, belongs p.s.v. $< \cdot |^{-}_{\parallel p \parallel}$.

Hence it follows that the tables in question similar to the tables of modular addition can be abbreviated/reduced four times, moreover work with these tables in form will be adequate to work with the tables of modular addition.

In appendix 2 are given the full/total/complete tables of recodings of the type

$$<-< w |^{-}_{q_k} |_{p_m,\ \overline{p}_m} \ (k < m) \ \begin{matrix} 1 \leqslant k \leqslant 6 \\ 2 \leqslant m \leqslant 6 \end{matrix}.$$

On the upper and lower sides of the intersection α of lines with β the column of tables are contained the leductions of a number $-<w|_{\bar{q}_k}$ on moduli/modules p_m and $\bar{p}_m$ respectively. The technical realization of similar tables is reduced to that so that each table of recodings of this type would be furnished with two decoders on moduli/modules p_k and $\bar{p}_k$.

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Let us now give the more detailed interpretation of the block diagram of Δ- algorithm in connection with the nonpositional numeration system with the pair-wise conjugated/combined bases/bases.

For this it suffices to decipher the assembly of the form, depicted in figure 45. But since the general/common/total structure of the described earlier diagram is retained, the mentioned assembly takes the form, shown in figure 46.

Let us note that during the tabular realization of modular arithmetic it is expedient the recoding, connected with the modular multiplication of the result of addition for constant $q_{k,m}^{-1}(\bar{q}_{k,m}^{-1})$, to provide in the table of modular addition, i.e., at the output/yield of module adder to put out the deduction, multiplied to the constant.

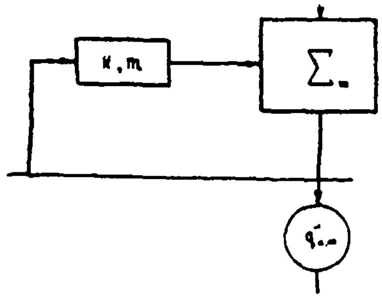

Fig. 45.

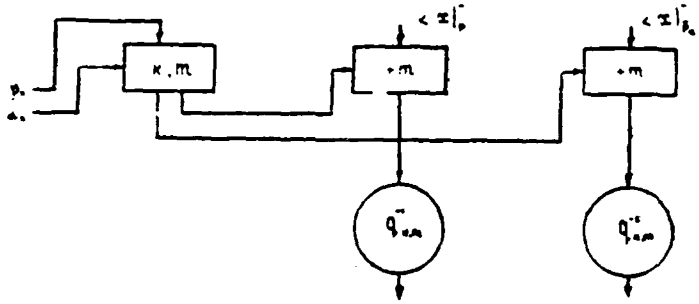

Fig. 46. $\boxed{m}$ - modular adder on $\bmod p_m$, $\boxed{\bar{m}}$ - modular adder on $\bmod \bar{p}_m$, $q^{-1}_{k,m}$ - recoding of type $<q_k^{-1}\zeta|_{\bar{p}_m}$, $\bar{q}^{-1}_{k,m}$ - recoding of type $<\bar{q}_k^{-1}\zeta|_{\bar{\bar{p}}_m}$, $\boxed{k,m}$ - recoding of type $<\omega(\alpha_k, \beta_k)|_{\bar{p}_m}, \bar{p}_m$.

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In conclusion let us present illustrative examples. Let us select as the basis of system pair-wise conjugated/combined ts.k.ch.

$$(2+3i)(2-3i)=13,\ (1+4i)(1-4i)=17,\ (3+4i)(3-4i)=25,$$
$$(2+5i)(2-5i)=29.$$

Example 1. Let us find components α of the mixed positional representation of ts.k.ch. A, preset by the nonpositional code, in

the canonical deductions

$$A \div (1-i,\ 1+8i,\ 2-3i,\ -7+6i).$$

We will usa Δ-algoritha

$$\begin{array}{llllll}
1-i & 1+8i & 2-3i & -7+6i & & \\
\downarrow & \underline{-8i} & \underline{1-2i} & \underline{-8+7i} & \sim\langle x+y \mid^{-}_{q_k} & (k=2,3,4) \\
\eta_1 & 2i & 2-4i & -14+5i & \sim\langle q_{1r}x \mid^{-}_{q_r} & (r=2,3,4) \\
 & \downarrow & \underline{2-6i} & \underline{-14+3i} & \sim\langle x+y \mid^{-}_{q_k} & (k=3,4) \\
 & \eta_2 & 6+7i & 6+7i & \sim\langle q_{2r}x \mid^{-}_{q_r} & (r=3,4) \\
 & & \downarrow & \underline{0} & & \\
 & & \eta_3 & \downarrow & & \\
 & & & \eta_4 & &
\end{array}$$

Hence we find tha unknown representation

$$A = (1-i) + 2i\cdot 13 + (6+7i)\,13\cdot 17 + 0\cdot 13\cdot 17\cdot 25.$$

Example 2. Ts.k.ch. A is preset in tha pair-wise conjugated/combined deductions

$$A \div (6,\ 9;\ 16,\ 3;\ 23,\ 6;\ 8,\ 7).$$

To fini its components ξ_i, η_i of the mixed positional representation. We will use the block diagcam of Δ-algoritha:

$$\begin{array}{lllll}
6,9 & 16,3 & 23,6 & 8,7 & \text{(1) перекодировка } (6,9)\to\langle -W \mid^{+}_{1\pm 4i,\ 3\pm 4i,\ 2\pm 5i} \\
\downarrow & \underline{8,12} & \underline{17,6} & \underline{16,11} & \\
\xi_1\eta_1 & 2,15 & 15,12 & 24,18 & \text{(2) модульная сумма, домноженная на константы } \langle q^{-1} \mid_{17,\ 25,\ 29} = (4,2,9) \\
 & 8,9 & 5,24 & 13,17 & \\
 & \downarrow & \underline{14,11} & \underline{24,5} & \text{(1) перекодировка } (8,9)\to\langle -W \mid^{+}_{3\pm 4i,\ 2\pm 5i} \\
 & \xi_2\eta_2 & 19,10 & 8,22 & \text{(2) модульная сумма, домноженная на константы } \langle q^{-1} \mid_{25,29} = (3,12) \\
 & & 7,5 & 9,3 & \\
 & & \downarrow & \underline{20,26} & \text{(1) перекодировка } (7,5)\to\langle -W \mid^{+}_{2\pm 5i} \\
 & & \xi_3\eta_3 & 00 & \text{(3) модульная сумма} \\
 & & & \downarrow & \\
 & & & \xi_4\eta_4 &
\end{array}$$

Key: (1). recoding. (2). modular sum, multiplied to constants. (3). modular sum.

§ 8. Operations of reduction and expanding the range.

Let be preset two groups of the pair-wise conjugated/combined mutually simple bases/bases. Each of these groups forms ranges (Q_n), (D_m), and in the set they form range $(Q_n D_m)$.

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With the work with the numbers, represented by the nonpositional code, frequently appear the following two tasks:

1) number $z \in (Q_n)$ is represented by remainders/residues on the bases/bases, which are determining range Q_n. It is necessary to determine the remainders/residues of number z on the bases/bases of range D_m;

2) number $z \in (Q_n D_m)$ is represented by remainders/residues on the bases/bases of range $(Q_n D_m)$. It is necessary to find the remainders/residues of number w of such, in order to

$$z = r + wD_m,$$

where

$$r \in (D_m).$$

From the property of the invariance of range $(Q_n D_m)$ it follows that $w \in (Q_n)$. The process of the solution of the first problem it is accepted to call the operation of expanding the range, and the second - by operation of range reduction $(Q_n D_m)$ to range (D_m).

Since for the realization of any of these operations is required to analyze the content bit one or the other set of bases/bases, they must be related to the bit of nonmodule operations.

In a sense these tasks are mutually-reverse, i.e., the algorithm of solution of one of them can be used for solving another.

Actually/really, let be known the algorithm of the operation of expansion, then for the realization of range reduction it is possible to act as follows.

It is obvious, the number, represented as remainders/residues on the bases/bases of range (D_m), is equal to r. Widening the nonpositional representation of this number to range (Q_n) and subtracting it from the nonpositional representation of number z in the range $(Q_n D_m)$, we will obtain number wD_m, whose nonpositional representation in the range $(Q_n D_m)$ in the bits according to the bases/bases of range (D_m) will contain zero. Multiplying this number on $< D_m^{-1} |_{Q_n}$, represented in the remainders/residues on the bases/bases of range (Q_n), we will obtain the unknown deductions of number w. Conversely, let be known the algorithm of range reduction and is ts.k.ch. $\overline{we(Q_n)}$, the given one by remainders/residues on the

bases/bases of range (Q_n). It is necessary to determine remainders/residues ts.k.ch. on the bases/bases of range (D_m).

We form number $w' \in (Q_n)$ such, in order to

$$<w' D_m \mid_{Q_n} = w.$$

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In the nonpositional numeration system of range (Q_n) this problem is solved by multiplication of ts.k.ch. w by constant $<D_m^{-1}|_{Q_n}$:

$$w' = <wD_m^{-1} \mid_{Q_n}.$$

In the range $(Q_n D_m)$ let us consider number $w' D_n$. Its nonpositional recording is such: in the bits along the bases/bases of range (Q_n) are placed the remainders/residues of number w, and in the bits on the bases/bases of range D_m - zero.

Using an algorithm of reduction, number $w' D_m$ can be presented in the form

$$w' D_m = w + rQ_n,$$

where ts.k.ch. r is preset by remainders/residues on the bases/bases of range (D_m).

From the latter/last equality it follows that

$$<w \mid_{D_m} = - < rQ_n \mid_{D_m}. \qquad (8.1)$$

Thereby the problem of expansion is solved. As it will be shown

below, the described operations play exceptionally important role in the implementation of such most important operations as the determination of the overflow of the sum of deductions, the operation of rounding.

Therefore the development of the efficient algorithms, which realize, let us say, the operation of expansion, composes the central task of the nonpositional numeration systems.

Are described below one of the algorithms (far the not most efficient), substantially using Δ- algorithm.

Algorithm of the operation of range reduction. Let be preset ts.k.ch. $z \in (Q_n D_m)$. It is necessary to determine remainders/residues of ts.k.ch. $w \in (Q_n)$ such, in order to

$$z = r + wD_m,$$

where

$$r \in (D_m). \qquad (8.2)$$

Applying Δ-algorithm taking into account the location of bases/bases $d_1, d_2, \ldots, d_m$ range (D_m), we will obtain first m of the digits of the polyadic representation of number z, i.e.

$$z = \zeta_1 + \zeta_2 d_1 + \ldots + \zeta_m d_1 d_2 \ldots d_{m-1}.$$

In view of the uniqueness of representation of ts.k.ch. z in the form (8.2), we consist that

$$w = \frac{z - (\zeta_1 + \zeta_2 d + \ldots + \zeta_m d_1 d_2 \ldots d_{m-1})}{D_m}. \qquad (8.3)$$

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Thus, the unknown deductions of number w can be obtained at the m step/pitch of Δ-algorithm, used to the nonpositional code ts.k.ch. z, preset by the remainders/residues

$$\underbrace{\overset{d_1}{\gamma_1, \delta_1}; \overset{d_2}{\gamma_2, \delta_2}; \ldots; \overset{d_m}{\gamma_m, \delta_m}}_{(D_m)}; \quad \underbrace{\overset{q_1}{\alpha_1, \beta_1}; \overset{q_2}{\alpha_2, \beta_2}; \ldots; \overset{q_n}{\alpha_n, \beta_n}}_{(Q_n)}.$$

The block diagram of this algorithm is given in figure 47.

Algorithm of the operation of expanding the range. As it follows from that presented, the operation of expansion (Fig. 48) can be realized on the basis of the algorithm of reduction. In the language of Δ-algorithm the diagram of expansion (Q_n) to range (D_m) simply coincides with the diagram of range reduction $(Q_n D_m)$ to range (Q_n). In this case at the initial moment of calculation in the bits along the bases/bases of range (D_m) are placed zero, and the final result, obtained as a result of transformations on the diagram of reduction to range (Q_n) is recoded in accordance with formula (8.1).

Observation. It should be noted that the property of the invariance of the polyadic codes relative to the ranges in question

allows for the operation of reduction and expanding the range to implement in the following more general/more common/more total setting:

a) to shorten range (Q_I), to range (Q_J);

b) to widen range (Q_J) to range (Q_I), where J - some subset of set I. I multitude of indices of bases/bases, which are determining range (Q_I).

Let us give illustrative examples.

Example 1. For the illustration of the operation of range reduction we will use example 2 of § 7.

Is reducible range on the basis $q_1=(2+3i)$ $(2-3i)$. $q_2=(1+4i)\times($ $1-4i)$. Then the humidly conjugated/combined code from full/total/complete range $Q_x = q_1$ q_2 q_3 q_4

$$(6,9;\ 16,3;\ 23,6;\ 8,7)$$

is converted:

$$\text{a)}\quad \overset{q_1}{} \quad (\overset{q_2}{8,9};\ \overset{q_3}{5,24};\ \overset{q_4}{13,17})$$

$$\text{b)}\quad \overset{q_1}{} \quad \overset{q_2}{} \quad (\overset{q_3}{7,5}\ \ \overset{q_4}{9,3})$$

Example 2. Is widened the range of the representation of number A:

$$A \leftarrow (\overset{q_1}{6,9}\ \ \overset{q_2}{16,3}\ \ \overset{q_3}{23,6}\ \ \overset{q_4}{?\ ?})$$

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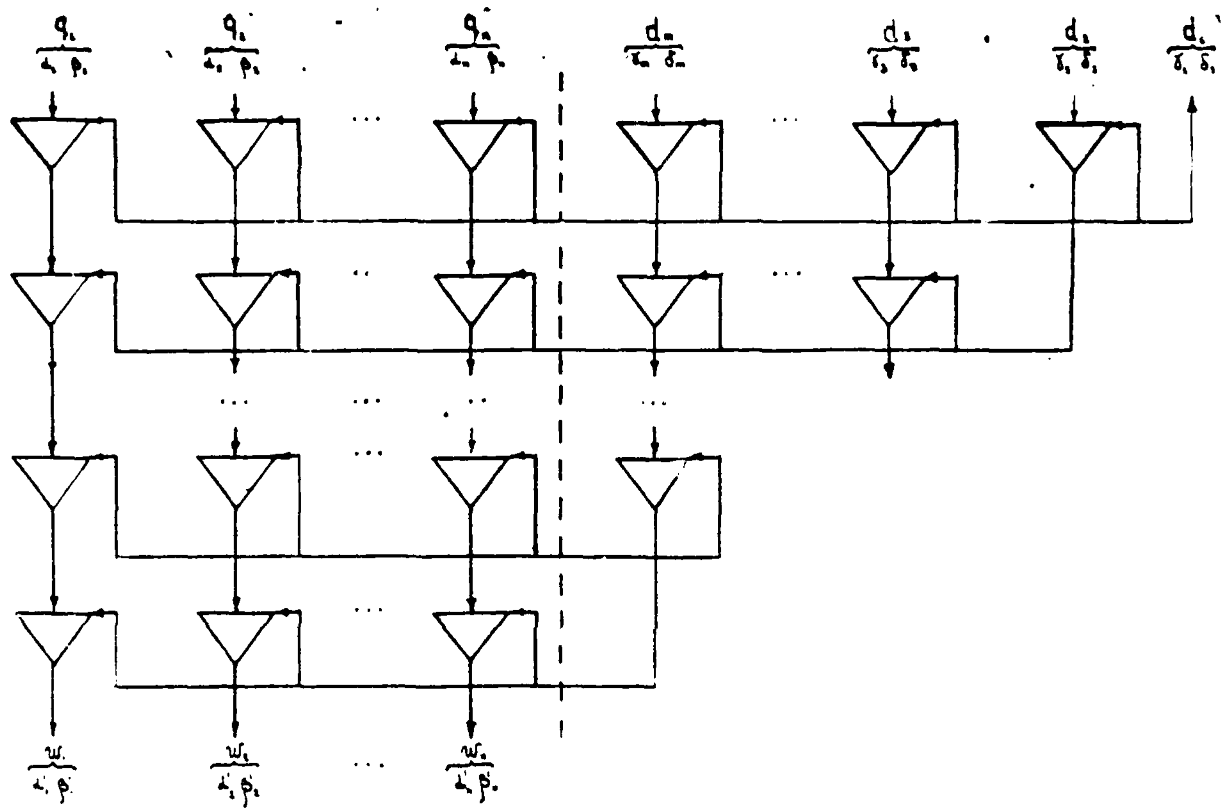

Fig. 47. Block liagram of algorithm of range reduction $(Q_n D_m)$ to range (D_m).

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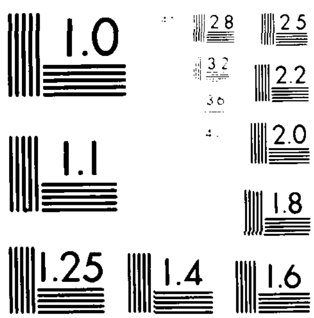

MICROCOPY RESOLUTION TEST CHART

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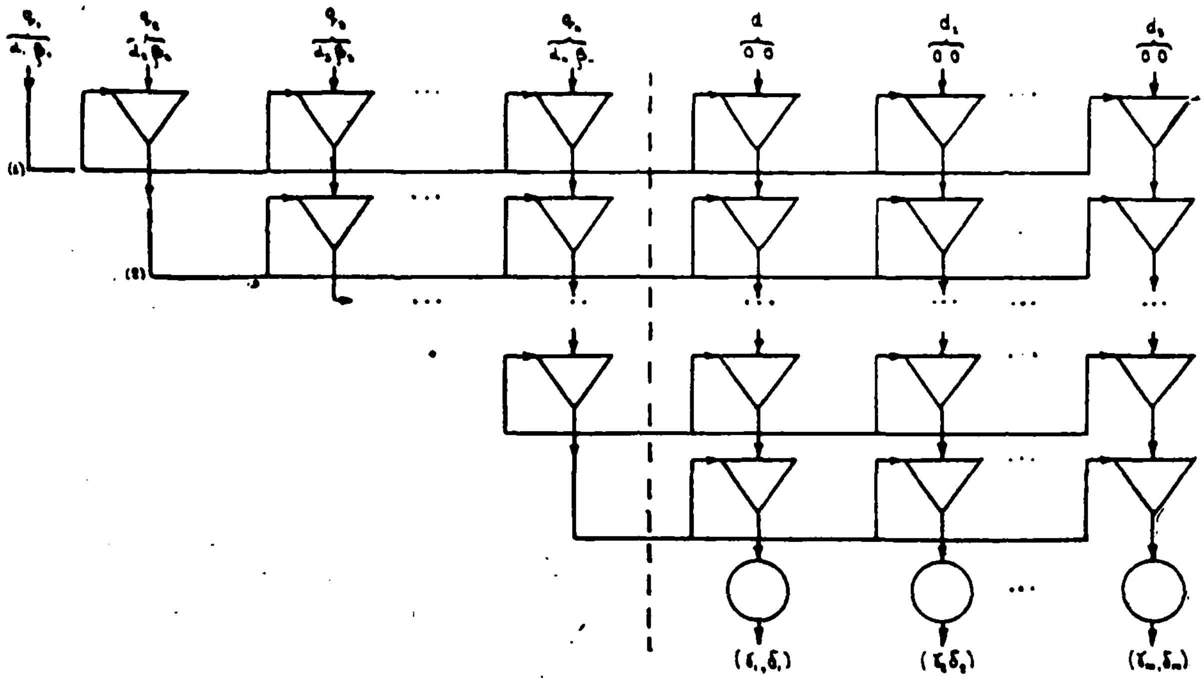

Fig. 48. Block diagram of algorithm of expansion of range (Q_n) to range (D_m).

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In accordance with the block diagram (Fig. 47) we have

q_1	q_2	q_3	q_4	
6,9	16,3	23,6	0,0	
	3,12	17,6	16,11	(1) перекодировка $(6,9) \to <-W|^{+}_{q_2, q_3, q_4}$
	2,15	15,12	16,11	(2) модульная сумма после умножения на кон-
	8,9	5,24	28,12	станты $<q^{-1}|_{17,25,29}=(4,2,9)$
		14,11	24,5	(1) перекодировка $(8,9) \to <-W|^{+}_{q_3, q_4}$
		19,10	23,17	(2) модульная сумма после умножения на кон-
		7,5	15,1	станты $<-q_1|_{25,29}=(3,12)$
			20,26	(1) перекодировка $(7,5) \to <-W|^{+}_{q_4}$
			6,27	(2) модульная сумма после умножения на кон-
			13,15	станты $<q^{-1}|_{29}=7$

(3) Искомые вычеты 8,7 (4) после перекодировки в соответствии (8,1) и инвертирования

Key: (1). recoding. (2). modular sum after multiplication by constants. (3). Unknown deductions. (4). after recoding in conformity (8.1) and inversion.

§ 9. Basic arithmetic operations over ts.k.ch. in the nonpositional numeration system.

Entire preceding material serves as basis for the construction of numeration system in the residual classes of complex numbers.

Let ts.k.ch. $Q_n = q_1 q_2 \dots q_n$ $(q_k = p_k \cdot \bar{p}_k)$ determine the range

$$(Q_n) = <\cdot|_{Q_n},$$

which we will call basic (or worker).

If ts.k.ch. $\overline{z \in (Q_n)}$, then it is unambiguously represented by the system of the remainders/residues

$$z \div (\alpha_1, \overset{q_1}{\beta_1}; \alpha_2, \overset{q_2}{\beta_2}; \dots; \alpha_n, \overset{q_n}{\beta_n}).$$

The operations of addition and multiplication above ts.k.ch. $z_1, z_2 \in (Q_n)$ are implemented modularly, i.e., if

$$z_1 \leftrightarrow (\alpha_1^{(1)}, \beta_1^{(1)}; \alpha_2^{(1)}, \beta_2^{(1)}; \ldots; \alpha_n^{(1)}, \beta_n^{(1)}),$$
$$z_2 \leftrightarrow (\alpha_1^{(2)}, \beta_1^{(2)}; \alpha_2^{(2)}, \beta_2^{(2)}; \ldots; \alpha_n^{(2)}, \beta_n^{(2)}),$$

then

$$z_1 + z_2 \leftrightarrow (\gamma_1, \delta_1; \gamma_2, \delta_2; \ldots; \gamma_n, \delta_n)$$

and

$$z_1 \cdot z_2 \leftrightarrow (\xi_1, \zeta_1; \xi_2, \zeta_2; \ldots; \xi_n, \zeta_n),$$

where

$$\gamma_k = <\alpha_k^{(1)} + \alpha_k^{(2)} \mid_{p_k}^{\pm}, \quad \delta_k = <\beta_k^{(1)} + \beta_k^{(2)} \mid_{p_k}^{\pm},$$
$$\xi_k = <\alpha_k^{(1)} \cdot \alpha_k^{(1)} \mid_{p_k}^{\pm}, \quad \zeta_k = <\beta_k^{(1)} \cdot \beta_k^{(2)} \mid_{p_k}^{\pm},$$

in this case as a result is obtained correct ts.k.ch., if result of operation will be found in the range (Q_n), if

$$z_1 + z_2 \in (Q_n) \text{ and } z_1 \cdot z_2 \in (Q_n).$$

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Otherwise the response/answer will differ from true in terms of multiple (Q_n).

Example. $Q_n = 13 \cdot 17 \cdot 25 \cdot 29.$

(1) Пусть $z_1 = 15+21i \leftrightarrow (1,2;\ 14,16;\ 18,12;\ 24,6)$
(2) и $z_2 = 8+6i \leftrightarrow (12,7;\ 10,18;\ 11,20,\ 18,17),$
(3) тогда $z_1 + z_2 \leftrightarrow (0,10;\ 7,12;\ 4,7,\ 18,23)$
(4) и $z_1 \cdot z_2 \leftrightarrow (12,8;\ 4,4;\ 23,15;\ 26,15).$

Key: (1). (2). and. (3). then.

It should be noted that for the modular arithmetic the characteristically in general following property:

if certain complicated operation is represented by the sequence of modular operations and in this case the result of complicated operation belongs to range (Q_n), then independent of possible outputs/yields for the range in the intermediate stages the result of complicated operation will be true.

When result of operation falls outside range, appears the need for restoring/reducing true result.

Usually this concerns the operation of addition. Relative to the operation of multiplication it is assumed that either the multipliers are scaled in such a way that does not appear the overflows for the basic range or there is surplus range over the basis, which makes it possible to maintain the correct result of product.

In the first case it is necessary to worry about the scaling of the cofactors before each multiplication, the secondly - operation of multiplication to accompany by the procedure of rounding. Both these of process by nature their are equivalent. As a rule, the first case

answers work TsVM [digital computer] in the so-called mode/conditions of fixed point, the secondly - in the mode/conditions of floating point. However, in the first case all possible overflows with the multiplication previously are considered and are eliminated by programmer by the corresponding scale factors, introduced into the program. In the second case this role automatically performs TsVM. In connection with which is required further time and equipment. Hence it becomes clear, that the best result on the high speed, obtained due to the deparallelization of arithmetic operations by the methods of the nonpositional numeration systems, should be expected in essence with the work in the mode/conditions of fixed point.

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Let us consider questions of overflow with the addition. Are possible two forms of nonpositional representation of ts.k.ch. depending on whether are chosen as p.s.v. the smallest or least positive residues. Let us consider at first the case of the smallest deductions. Range is determined by set ts.k.ch. x+iy, real and alleged parts of which satisfy the condition

$$0 < x < Q_n,$$
$$0 < y < Q_n.$$

Here in order to contain the symmetry of the integral network of composite plane, it is necessary to introduce concept the composite

sign of a number, which is represented by one of the dividers/denominators of unity. Then complex integers are registered in the form

$$z = i^{\alpha} w, \text{ where } w \in (Q_n),$$

which we will call the sign form of the representation of complex numbers. In this case is complicated the operation of addition, since appears the need for considering sign situations.

Let us pause in greater detail on this question.

Let be preset two ts.k.ch.

$$z_1 = i^{\alpha_1} w_1, \; z_2 = i^{\alpha_2} w_2, \; (w_1, \; w_2 \in Q_n),$$

moreover $\alpha_1 \neq \alpha_2$, then

$$z_1 + z_2 = i^{\alpha_1} (w_1 + i^{\beta} w_2), \tag{9.1}$$

where

$$\beta = | \alpha_2 - \alpha_1 |_4^+.$$

Let $w_2 = A_2 + iB_2$ (A_2, $B_2 \geqslant 0$), then be obvious,

$$< i^{\omega} (A_2 + iB_2) |_{Q_n}^{\pm} = \begin{cases} A_2 + iB_2, & \text{если}^{(1)} \; \omega = 0 \\ Q_n - B_2 + iA_2, & \text{если}^{(1)} \; \omega = 1 \\ (Q_n - A_2) + i (Q_n - B_2), & \text{если}^{(1)} \; \omega = 2 \\ B_2 + i (Q_n - A_2), & \text{если}^{(1)} \; \omega = 3. \end{cases}$$

Key: (1). if.

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In the remainders/residues these numbers respectively take the form

$$< i^{\varphi}(A_2+iB_2)\,|^{+}_{p_k} = < i^{\varphi}\alpha_k\,|^{+}_{p_k},$$
$$< i^{\varphi}(A_2+iB_2)\,|^{\pm}_{p_k} = < i^{\varphi}\beta_k\,|^{\pm}_{p_k},$$

i.e., multiplication of ts.k.ch. w on the degree of the imaginary unit in the nonpositional representation they answer three types of recodings. With the use of an isomorphism of composite deductions to real deductions these recodings take the form: if α=x+iy, then

$$< i^{\varphi}\alpha_k\,|^{+}_{p_k} <\!\!=\!\!> |\,\rho_k^{\varphi}(x+\rho_k y)\,|^{\pm}_{\epsilon_k} <\!\!=\!\!> < \alpha_k'\,|^{+}_{p_k},$$

where ρ_k - coefficient of isomorphism, which corresponds to modulus/module p_k. Let now $w_i=A_i+iB_i$ (A_i, $B_i \geq 0$), then, being returned to sum (9.1), we note that are possible the following situations:

$$w_1+i^{\varphi}w_2 = A_1+iB_1+\begin{cases} -B_2+A_2 i, & \text{если } \varphi=1 \\ -A_2-B_2 i, & \text{если } \varphi=2 \\ B_2-A_2 i, & \text{если } \varphi=3. \end{cases}$$

Key: (1). if.

The result of this operation will be registered in the sign form and will be indicated the value of overflow for range $(2Q_n)^-$, if overflow occurred.

Case φ=1. to nonpositional exponent arithmetic - B_2 is represented by number $B_2'=Q_n-B_2$, therefore we will have

$$(A_1+iB_1)+(B_2'+iA_2)=(A_1+B_2')+i(B_1+A_2)=$$
$$=C+Di+\eta Q_n, \qquad (9.2)$$

where C, D>0.

The value of overflow η can take the values:

$$\eta=\begin{cases}0\\1\\i\\1+i.\end{cases}$$

It should be noted that value η is not erect image of the actual overflow of the real complex quantities, represented by the nonpositional code in the sign form.

Value η, besides information about the possible actual overflow, contains also information about the available sign combinations. In connection with this should be analyzed all possible combinations.

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3) Let $\eta=0$. This means that actually occurred situation $A_1<B_2$ and $0\leqslant B_1+A_2<Q_n$, true sum (with $\beta=1$) takes the form

$$w_1+iw_2=-C'+Di, \text{ where } C'=Q_n-C.$$

Let us register the obtained sum in the sign form -

$C'+Di=i\overbrace{(-i)(-C'+D'i)}=i(D+C'i)$.

Result can be reduced to the following table.

η	(1) Переполнение	(2) Перекодировка модульной суммы	(3) Знак истинной суммы
$\varphi=1,\ \eta=0$	(4) Нет	(5) Умножить на $(+i^3)$	i

Key: (1). Overfilling. (2). Recoding of modular sum. (3). Sign of true sum. (4). No. (5). To multiply on $(+i^3)$.

Similarly can be traced all remaining situations. It remains to consider the algorithm of the determination of overflow for the numbers, represented by the nonpositional code. From the relationship/ratio of type (9.2) for the arbitrary ω follows that for the catching of overflow through range $(\dot{Q}_n)$ is necessary the redundancy, determined by modulus/module 2.

Thus, in the nonpositional representation of number $z^* \in (Q_n)$ must be present surplus composite deduction of this number on mod 2. Let us designate it through ε, i.e. $< z |_2^+ = \varepsilon$.

Let

$$< w_1 |_2^+ = \varepsilon_1,$$

$$\ll i^\alpha w_2 |_{Q_n}^+ |_2^+ = \varepsilon_2,$$

then, passing from equality (9.2) to the comparison on mod of 2, we

obtain

$$<\varepsilon_1+\varepsilon_2|_2^+ = \ll C+Di|_2^+ + <\eta Q_n|_2^+|_2^+.$$

Hence it follows that

$$\eta = <Q_n^{-1}(<\varepsilon_1+\varepsilon_2|_2^+ - <C+Di|_2^+)|_2^+. \qquad (9.3)$$

In the latter/last formula figures the ieduction of the modular sum C+Di on mod 2. It can be obtained only by the expansion of the nonpositional representation of sum $C+Di\varepsilon(D_n)$ to surplus basis/base 2.

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Table of overflows.

η	(1) Переполнение	(2) Величина переполнения	(3) Перекодировка модульной суммы	(4) Знак истин. суммы
		$\omega=0$		
0	(4a) Нет	0	(5) Остается неизменной	+1
1	(6) По действительной части	1	(5) Остается неизменной	+1
i	(7) По мнимой части	i	(5) Остается неизменной	+1
$1+i$	(8) По действительной и мнимой частям	$1+i$	(5) Остается неизменной	+1
		$\omega=1$		
0	(4a) Нет	0	(9) Умножить на i^3	i
1	»	0	(5) Остается неизменной	+1
i	(7) По мнимой части	i	(9) Умножить на i^3	i
$1+i$	(7) По мнимой части	i	(5) Остается неизменной	+1
		$\omega=2$		
0	(4a) Нет	0	(9) Умножить на i^2	i^2
1	»	0	(9) Умножить на i	i^3
i	»	0	(9) Умножить на i^3	i
$1+i$	»	0	(5) Остается неизменной	+1
		$\omega=3$		
0	(4a) Нет	0	(9) Умножить на i	i^3
1	(6) По действительной части	1	(9) Умножить на i	i^3
i	(4a) Нет	0	(5) Остается неизменной	+1
$+1+i$	(6) По действительной части	1	(5) Остается неизменной	+1

Key: (1). Overflow. (2). Value of overflow. (3). Recoding of modular sum. (4). Sign of truths. of sum. (4a). No. (5). It remains constant/invariable. (6). On real part. (7). On imaginary part. (8). On real and alleged parts. (9). To multiply on.

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Thus, any checking to the overflow of addition requires the execution of the nonmodule operation of expansion, and the algorithm of the determination of overflow is formed/shaped in accordance with equality (9.3).

With the work in the mode/conditions of floating point usually it is necessary to know the value of actual overflow, and result to present in the form accepted. In connection with this let us consider the case when appears actual overflow.

Case $\omega=0$ and $\eta_1=1$. On the basis of the relationship/ratio of type (9.2), we see that the overflow occurs in the real part of the sum, in this case value η is the actual value of overflow for the sum of form (9.2).

Case $\omega=1$ and $\eta_1=\bar{i}$. In accordance with the previously case $\omega=1$ and

$\eta=0$ examined we will have

$$w_1+iw_2=i(D+C'i)+iQ_n,$$

i.e., value $\eta=i$ is here again the value of actual overflow.

A similar analysis of all possible situations leads to the table of overflows (see page 180; one should again recall that the table concerns onl the sums of form $w_1+i^* w_2$, where $w_1, w_2 \in (D_n)$).The major advantage of work with the least non-negative residue consists only of a smaller quantity of samples of overflow with the addition, which undoubtedly creates some conveniences for the engineering realization of the corresponding devices/equipment on two-discrete elements/cells.

In order to avoid the analysis of sign situations with the addition and at the same time to preserve the advantages of the smallest non-negative deductions, it is possible, similar this is done in the class ts.k.ch., to introduce the concept of "artificial sign". However, with this appears the need for the introduction of further procedures to the processes of rounding in order to preserve the artificial sign of the unknown result. These questions here will not be detailed.

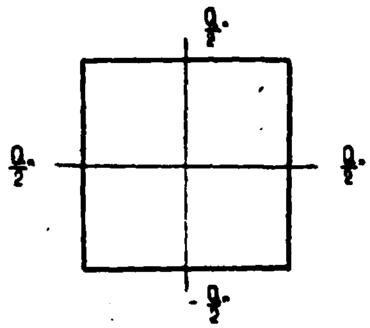

Fig. 49. Range (Q_n).

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Considerable advantage in the implementation of many most important operations of composite arithmetic, such, as multiplication by the divider/denominator of unity, the operation of composite coupling, the absence of the need for analyzing sign with ts.k.ch. with addition, etc., give transition to the examination of remainders/residues in the class of a.n.v. However, a quantity of samples of overfilling with the addition here increases doubly against the case of p.s.v. It is necessary to note that for the tabular realization of modular operations this fact does not have vital importance.

Thus, let us pass to the examination of the case of the least positive residues and subsequently all arithmetic operations we will

examine in the class of the least positive residues (Fig. 49).

In the class of a.n.v. "composite sign" ts.k.ch. is inscribed in the nonpositional recording of a number and in the quite nonpositional arithmetic interest us will not be.

Let

$$z \in < \cdot \mid \overline{\ }_{Q_n}$$

and

$$z \div (\alpha_1, \beta_1; \alpha_2, \beta_2; \dots; \alpha_n, \beta_n),$$

where

$$\alpha_k \in < \cdot \mid \overline{\ }_{p_k}, \ \beta_k \in < \cdot \mid \overline{\overline{\ }}_{p_k} \quad (1 \leqslant k \leqslant n).$$

It is obvious that for determining the composite sign of a number, preset by the nonpositional code, it is necessary to translate it into the positional code, for example, polyadic. Actually/really,

$$z \in (Q_n)$$

and

$$z = \zeta_1 + \zeta_2 q_1 + \zeta_3 q_1 q_2 + \dots + \zeta_m q_1 \dots q_{m-1}, \ (m < n), \quad (9.4)$$

where

$$\zeta_k \in < \cdot \mid \overline{\ }_{q_k},$$

moreover

$$\zeta_m \neq 0.$$

Let it be further

$$z = A' + B' i$$

and

$$\zeta_k = x_k + iy_k, \quad 1 < k < m,$$

then

$$A' = \sum_{k=1}^{m} x_k q_1 q_2 \ldots q_{k-1}$$
$$x_k,\ y_k \in | \cdot |_{q_k}^{-}.$$
$$B' = \sum_{k=1}^{m} y_k q_1 q_2 \ldots q_{k-1}$$

Page 183. Hence it follows that if $x_m, y_m \neq 0$, then the signs of numbers A' and B' are determined by the signs of numbers x_m and y_m respectively.

Consequently, the composite sign of number z is determined by the composite sign of the more significant digit of the polyadic disintegration of number z.

Let us consider the case when $x_m \cdot y_m = 0$. Let $x_m \neq 0$ and $y_m = 0$. Then for determining the sign of number B' we must know the sign of the senior, different from zero, the digit of the disintegration of real number B' into the polyadic code. Hence it follows that the information about the sign ts.k.ch. z can be obtained from the expansion of ts.k.ch. z into the polyadic code, since the combination of the signs of numbers A', B' uniquely determines sign of ts.k.ch.

A'+B'i.

R some operations, characteristic for the arithmetics of ts.k.ch., which do not derive/conclude number z for range (Q_n). A number of such operations includes the multiplication of number z by the divider/denominator of unity.

• If

$$z \leftrightarrow (\alpha_1, \beta_1; \alpha_2, \beta_2; \ldots; \alpha_n, \beta_n) \in (Q_n),$$

then

$$i^\omega z = (i^\omega \alpha_1, i^\omega \beta_1; i^\omega \alpha_2, i^\omega \beta_2; \ldots; i^\omega \alpha_n, i^\omega \beta_n) \in (Q_n).$$

Taking into account that a.n.v. they represent by the code

$$\alpha_k = n_k \xi_k, \ \beta_k = m_k \eta_k,$$

deductions $i^\omega \alpha_k, i^\omega \beta_k$ take the form

$$i^\omega \alpha_k = | n_k + \omega |_4^+ \xi_k, \ i^\omega \beta_k = | m_k + \omega |_4^+ \eta_k,$$

i.e., multiplication of ts.k.ch. z on i^ω is reduced to the appropriate change in the orders of deductions, their mantissa in this case they remain constant/invariable.

Jump operation to the conjugated/combined number. On the remainders/residues of ts.k.ch. z it is necessary to construct remainders/residues of ts.k.ch. $\bar{z}$. Let us consider canonical deduction ts.k.ch. z:

$$w_k = \alpha_k m_k \bar{p}_k + \beta_k \bar{m}_k p_k - r_k q_k.$$

Hence

$$\bar{w}_k = \bar{\alpha}_k \bar{m}_k p_k + \bar{\beta}_k m_k p_k - \bar{r}_k q_k,$$

i.e., the pair of remainders/residues α_k, β_k, which is determining canonical deduction, with the jump operation to the conjugated/combined number is converted into pair $(\bar{\beta}_k, \bar{\alpha}_k)$. In other words,

$$\overline{(\alpha_k, \beta_k)} = (\bar{\beta}_k, \bar{\alpha}_k).$$

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This transformation requires the operation of recoding. The function of recoding can be determined, on the basis of the principle of isomorphism. Let us consider, for example, the function of recoding of basis/base $\bar{p}_k$. From

$$x_k + iy_k <==> | x_k + \rho_k y_k |^+_{q_k},$$

where ρ_k - coefficient of isomorphism, which corresponds to basis/base p_k, it follows that

$$x_k - iy_k <==> | x_k - \rho_k y_k |^+_{q_k} = | x_k + \bar{\rho}_k y_k |^+_{q_k},$$

where $\bar{\rho}_k$ - coefficient of isomorphism, which corresponds to basis/base $\bar{p}_k$.

Algorithm of the determination of overflow with the addition in the class of a.n.v. Since overflow with the addition in the class of a.n.v. is characterized p.s.a.n.v. on mod 3, than for determining the overflow through range (Q_n) it suffices to have surplus basis/base,

equal to 3.

Let $z_1, z_2 \in \langle \cdot |_{Q_n}^{-}$, then

$$z_1 + z_2 = w + \eta \cdot Q_n, \qquad (9.5)$$

where

$$w \in \langle \cdot |_{Q_n}^{-}, \ \eta \in \langle \cdot |_3^{-}.$$

Passing in equality (9.5) to the comparison on mod of 3, we will obtain

$$s' = \langle\langle z_1 |_3^{-} + \langle z_2 |_3^{-} |_3^{-} = \langle\langle w |_3^{-} + \eta \langle Q |_3^{-} |_3^{-}.$$

Hence the value of overflow is determined from the formula

$$\eta = \langle\langle Q_n^{-1} |_3^{-} (s' - \langle w |_3^{-}) |_3^{-}.$$

Deduction $\langle w |_3^{-}$ is obtained by the expansion of the representation of modular sum to basis/base mod 3.

The block diagram of this algorithm is shown in figure 50.

Example. Basic range of system $Q_n = 13 \cdot 17 \cdot 25$.

Constants: $\langle Q_n^{-1} |_3^{-} = -1; \ -13 \cdot 17 \cdot 25 \ (\mathrm{mod}\ 3) = -2,$

$\langle 13^{-1} |_{17,25,3} = (4, 2, 1),$

$\langle 17^{-1} |_{25,3} = (3, 2)$

$\langle 25^{-1} |_3 = (1).$

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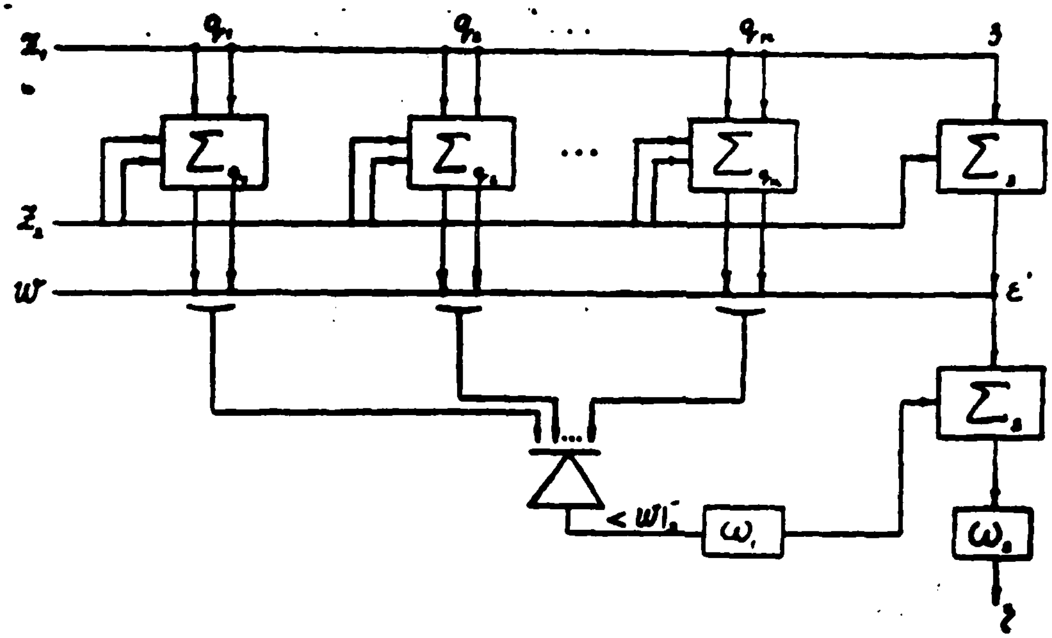

Fig. 50. Block liagram of algorithm of determination of value of overflow.

$\overline{\triangle}$ - operation of expansion,

$\boxed{\omega_1}$ - the operation of the recoding of deduction $<w|_s^-$ by the deduction of opposite sign,

$\boxed{\omega_2}$ - the operation of the multiplication of deduction to constant $<Q_n^{-1}|_s^-$; at the output/yield of this block is obtained the value of overflow.

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To determine to re-divide overflow with the addition of the numbers

$$
\begin{array}{lllll}
 & 13 & 17 & 25 & 3 \\
z_1 \leftarrow (7,10; & 0,8\ ; & 15,9; & -1+i) \\
z_2 \leftarrow (6,3\ ; & 5,11; & 24,5; & 1+i)
\end{array}
$$

According to block diagram (Fig. 48), we have:

$$
\begin{array}{lllll}
z_1 \leftarrow 7,10; & 0,8; & 15,9; & -1+i & \\
z_2 \leftarrow 6,3; & 5,11; & 24,5; & 1+i & \\
\hline
W \leftarrow 0,0; & 5,[illegible]; & 14,14; & -i & \sim \epsilon' \\
0,0; & 4,[illegible] & 14,14; & 0 & \\
 & 8,8; & 3,3; & 0 & \\
 & & 4,2; & i & \\
\cline{3-4}
 & & 7,5; & i & \sim \bar{\Delta} \\
 & & 21,15; & -i & \\
 & & & 1+i & \\
\cline{4-4}
 & & & 1 & \\
 & & & & \\
 & & 1\cdot(-2)=1 & & \sim \ <W \mid_{\bar{3}} \\
 & & -1 & & \sim -<W \mid_{\bar{3}}
\end{array}
$$

final overflow is such:

$$\eta = [\epsilon' + (-<w \mid_{\bar{3}})] < Q_n^{-1} \mid_{\bar{3}} = (-i-1)(-1) = 1+i.$$

In conclusion let us present one translation algorithm of nonpositional representation of complex numbers into the positional.

Let $z \in <\cdot \mid_{\theta_n}$.

It is necessary to determine the digits of the positional representation of the number

$$z = \zeta_0 + \zeta_1 q + \zeta_2 q^2 + \ldots + \zeta_m q^m, \qquad (9.6)$$

where

$$\zeta_0 \in <\cdot|_q^-.$$

For this purpose it is necessary to assume that in the bases/bases of range (Q_n) is included basis/base q. In this case, obviously, the deduction of ts.k.ch. z on basis/base q determines the low-order digit ζ_0 of representation (9.6). For determining the following digit ζ_1 we convert (9.6) to the form

$$\frac{z-z_0}{q} = \zeta_1 + \zeta_2 q^2 + \ldots + \zeta_m q^m,$$

i.e.

$$\zeta_1 = <\frac{z-z_0}{q}\Big|_q^-.$$

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In the language of the nonpositional representation of numbers this operation appears as follows:

1) from number z, preset by the nonpositional code, is subtracted ts.k.ch. ζ_0, represented by the nonpositional code on all bases/bases of range (Q_n);

2) the obtained number is divided into q, this operation is implemented by the modular multiplication of the result of the previous operation on ts.k.ch.:

$$<q^{-1}|_{Q_n,q}^- = (<q^{-1}|_{p_1}^-, <q^{-1}|_{p_2}^-; \ldots; <q^{-1}|_{p_{n'}}^-, <q^{-1}|_{p_n}^-).$$

As a result we will obtain the number

$$< q^{-1}(z - \bar{z}_0) \mid \bar{}_{Q_n q},$$

represented by its remainders/residues in the range Q_n/q:

3) let us expand this representation to base q. The obtained deduction on basis/base q will determine digit ζ.

Continuing this process consecutively/serially, we will obtain all digits of representation (9.6).

The block diagram of the described operation is shown in figure 51.

Example. Transfer number A-(4, 11; 5, 3; 1, 23; 3, 1) from the nonpositional system with the pair-wise conjugated/combined bases/bases $13=(2+3i)\cdot(2-3i)$, $17=(1+4i)\cdot(1-4i)$, $29=(2+5i)\cdot(2-5i)$, $10=(1+3i)\cdot(1-3i)$ into the ordinary decimal system. The composite digit of unity of decimal representation of a number, obviously, corresponds to deductions on the bases/bases $10=(1+3i)\ (1-3i)$, i.e.

$$< 3,1 \mid {}^{+}_{10} = 7 + 2i.$$

Composite digit with first degree of 10 is determined according to block diagram (Fig. 49):

13	17	29	10	
4.11	5.3	1.23	3.1	
+ 3.9	2.1	27.17		(1) перекодировка $<-(3,1)\mid^{-}_{2=3i,\ 1=4i,\ 2=5i}$
7.7	7.4	28.11		(2) сумма
2.2	16.14	26.4		(3) код, полученный в результате сокращения диапазона на основание 10
	+ 15.15	27.27	8.8	(1) перекодировка $<-2\mid^{+}_{1=4i,\ 2=5i,\ 1=3i}$
	14.12	24.2	8.8	(2) сумма
	5.14	13.18	6.6	(1) перекодировка $<(z-\zeta_0)q^{-1}\mid Q_n\, q$
		+ 16.11	6.2	(1) перекодировка $<-(5,14)\mid^{-}_{2=5i,\ 1=3i}$
		0.0	2.8	(2) сумма
			6.4	(1) перекодировка $<(z-\zeta)\, q^{-1}\mid Q_n\, q$

$13\cdot 17\equiv 1 \pmod{10}$

$1\cdot(6.4)\equiv 6.4 \pmod{10}$

$10-6=4$

$10-4=6$

(4) Итак, искомое число: $<4,6\mid^{+}_{10}=5+3i.$

Key: (1). recoding. (2). sum. (3). code, obtained as a result of range reduction to basis/base 10. (4). Thus, unknown quantity.

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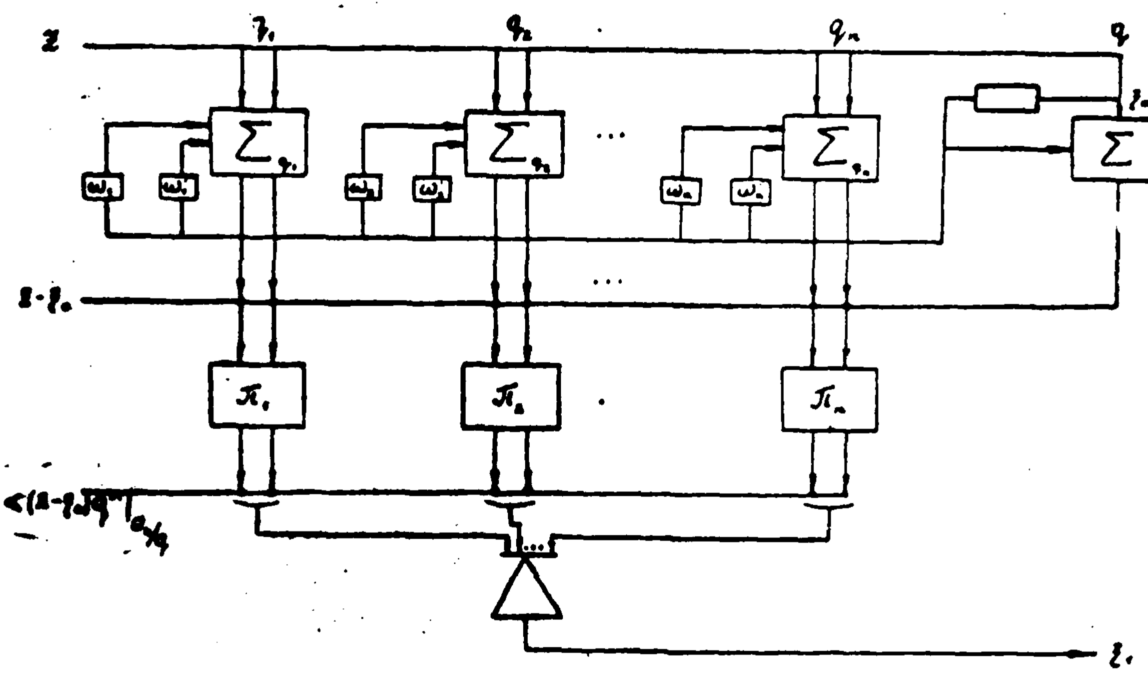

Fig. 51. Block diagram of algorithm of determination of digit of positional representation.

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Let us determine composite digit with second degree of 10. The initial code it will be: 2.2 16.14 28.4 4.6

13	17	29	10	
2,2	16,14	26,4	4,6	(1)
+10,6	0,7	17,2		перекодировка $<-(4,6)\mid_{13,\ 17,\ 29}$
12,8	16,4	14,6		(2)
9,6	5,14	13,18		код после сокращения на основание 10
	+12,3	16,11	6,2	(1) перекодировка $<-(9,6)\mid_{17,\ 29,\ 10}$
	0,0	0,0	6,2	
			2,4	(1) перекодировка $<(z-z_0)\,q^{-1}\mid Q_n\mid q$

$13\equiv 3 \pmod{10}$
$3,(2,4)\equiv 6,2 \pmod{10}$
$10-6=4$
$10-2=8$

Key: (1). recoding. (2). code after reduction to basis/base 10.

Therefore the unknown digit is equal to $<4,8\mid_{10} = 1+i$.

Thus, unknown quantity is equal

$$(1+i)\cdot 10^2 + (5+3i)\cdot 10^1 + (7+2i)\cdot 10^0 = 157+132i.$$

Actually/really,

$$157+132i \rightarrow (1 \overset{13}{+} 2i,\ 4 \overset{17}{-} 4i,\ 12 \overset{29}{-} 13i,\ 7 \overset{10}{+} 2i) = (4,11;\ 5,3;\ 1,23;\ 3,1).$$

§ 10. Arithmetic of fractional complex numbers.

The set of the fractions of the form

$$\alpha = \frac{z}{Q_n}, \qquad (10.1)$$

where ts.k.ch. $z<\cdot\mid\overline{Q}_n$ is formed the range of the fractions, subordinated to the limitation

$$-\frac{1}{2} < \mathrm{Re}\,\alpha,\ \mathrm{Im}\,\alpha < \frac{1}{2}.$$

Numerator z of each such fraction as ts.k.ch., belonging to range

(Q_n), let us unambiguously represent in absolutely smallest remainders/residues on the bases/bases of range Q_n.

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Since

$$a_1 + a_2 = \frac{z_1}{Q_n} + \frac{z_2}{Q_n} = \frac{z_1 + z_2}{Q_n},$$

$$t^m a = \frac{t^m z}{Q_n},$$

$$\bar{a} = \frac{\bar{z}}{Q_n},$$

then all described above algorithms apply to operations with the fractions of form (10.1).

Operation of multiplication. We have

$$a_1 \cdot a_2 = \frac{z_1}{Q_n} \cdot \frac{z_2}{Q_n} = \frac{z_1 \cdot z_2}{Q_n^2}. \tag{10.2}$$

Let

$$z_1 \cdot z_2 = \zeta + wQ_n, \tag{10.3}$$

where

$$\zeta \in (Q_n). \tag{10.4}$$

From (10.3) and (10.4) on the strength of the fact that Q_n - real number, follows that $w \in (Q_n)$.

Hence (10.2) it is possible to rewrite in the form

$$a_1 \cdot a_2 = \frac{\zeta}{Q_n^2} + \frac{w}{Q_n}$$

or

$$\left| a_1 \cdot a_2 - \frac{w}{Q_n} \right|^2 \leqslant \frac{1}{2Q_n^2}.$$

Latter/last evaluation indicates magnitude of error which appears during the replacement of fraction $\frac{z_1 \cdot z_2}{Q_n}$ to the fraction of form (10.1) $\frac{w}{Q_n}$.

Thus, the operation of the multiplication of fractions must contain the procedure of rounding which is expressed disregarding by value ζ of relationship/ratio (10.3). Let us paraphrase now these calculations for the language of residual classes.

In order to hold down/retain the value of the product of two arbitrary ts.k.ch. range (Q_n), are necessary certain surplus range (D_m), such that

$$(Q_n) \in (D_m).$$

Let us assume that such a surplus range (D_m), formed by bases/bases $D_m = d_1 d_2 \ldots d_m$, is, multipliers z_1 and z_2 are represented by remainders/residues in the range $(Q_n D_m)$. Then their modular product will determine the true value of product $z_1 \cdot z_2$, since $z_1 z_2 \in (D_m)$. From (10.3) it follows that

$$\zeta = < z_1 \cdot z_2 \mid_{\bar{Q}_n},$$

i.e., ζ is represented by the remainders/residues of product $z_1 \cdot z_2$ on the bases/bases of range Q_n. Widening the representation of a number

ς to the basis/base (D_m) and subtracting it from the nonpositional representation of number $z_1 \cdot z_2$, we will obtain number $w \cdot Q_n$, represented by remainders/residues on the bases/bases of range ($Q_n D_m$), where in the bits on the bases/bases of range (Q_n) are arranged/located zero.

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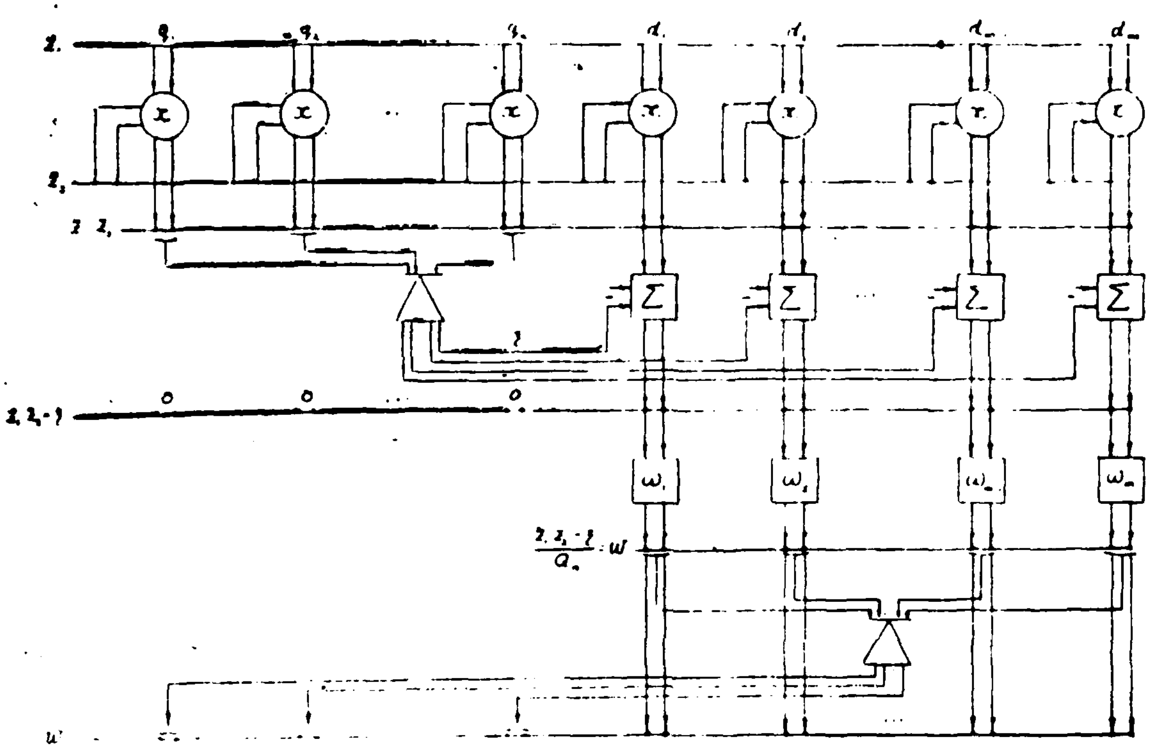

Fig. 52. Block diagram of multiplication of fractions.

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Reducing this number on Q_n, we will obtain number w, represented by remainders/residues on the bases/bases of range D_m. Widening the nonpositional representation of number w to range Q_n, we will obtain the expression of the unknown value w in the remainders/residues on the bases/bases of range $(Q_n D_m)$. By this, strictly, and is completed

the algorithm of the multiplication of the fractions, accompanied by the procedure of rounding. The block diagram of this algorithm is given in figure 52. It is assumed that the numerators of fractions of form (10.1) are represented on the bases/bases of range $(Q_n D_m)$.

Observation. As is known, in the positional arithmetic basic production time of multiplication is absorbed by the very procedure of multiplication. The process of rounding is here implemented automatically. For the nonpositional arithmetic in the residual classes the picture is opposite.

The procedure of multiplication is implemented in the minimum time (for one modular stroke/cycle of multiplication), whereas basic time by the fulfillment of the entire operation of multiplication as a whole occupies the procedure, connected with the need for the rounding of result. Since the fundamental principle of deparallelization requires reduction to the minimum of a number of nonmodule operations, then maximum prize in the time in the nonpositional arithmetic of residual classes can be achieved/reached, for example, during the calculations, which consist of a large number of sums of the products

$$\sum_{i=1}^{N} a_i b_i,$$

since the process of calculations here can be organized so as to modularly store modular products, and in the completion of this operation to only pass to the procedure of rounding.

ACHMENT 1

LES FOR CONVERSION OF CANONICAL RESIDUES INTO RESIDUES BY MODULE

mod $2+3i$

A \ B	6	5	4	3	2	1	0	−1	−2	−3	−4	−5	−6
6	0.2	1.3	2.1	2.3	0.3	0.1	3.3	2.2	1.2	1.1	0.0	3.1	3.2
5	0.1	3.3	2.2	1.2	1.1	0.0	3.1	3.2	0.2	1.3	2.1	2.3	0.3
4	0.0	3.1	3.2	0.2	1.3	2.1	2.3	0.3	0.1	3.3	2.2	1.2	1.1
3	2.1	2.3	0.3	0.1	3.3	2.2	1.2	1.1	0.0	3.1	3.2	0.2	1.3
2	2.2	1.2	1.1	0.0	3.1	3.2	0.2	1.3	2.1	2.3	0.3	0.1	3.3
1	3.2	0.2	1.3	2.1	2.3	0.3	0.1	3.3	2.2	1.2	1.1	0.0	3.1
0	0.3	0.1	3.3	2.2	1.2	1.1	0.0	3.1	3.2	0.2	1.3	2.1	2.3
−1	1.1	0.0	3.1	3.2	0.2	1.3	2.1	2.3	0.3	0.1	3.3	2.2	1.2
−2	1.3	2.1	2.3	0.3	0.1	3.3	2.2	1.2	1.1	0.0	3.1	3.2	0.2
−3	3.3	2.2	1.2	1.1	0.0	3.1	3.2	0.2	1.3	2.1	2.3	0.3	0.1
−4	3.1	3.2	0.2	1.3	2.1	2.3	0.3	0.1	3.3	2.2	1.2	1.1	0.0
−5	2.3	0.3	0.1	3.3	2.2	1.2	1.1	0.0	3.1	3.2	0.2	1.3	2.1
−6	1.2	1.1	0.0	3.1	3.2	0.2	1.3	2.1	2.3	0.3	0.1	3.3	2.2

mod $2-3i$

A \ B	6	5	4	3	2	1	0	−1	−2	−3	−4	−5	−6
6	1.2	1.1	0.0	3.1	3.2	2.2	0.3	0.1	3.3	1.3	2.1	2.3	0.2
5	3.3	1.3	2.1	2.3	0.2	1.2	1.1	0.0	3.1	3.2	2.2	0.3	0.1
4	3.1	3.2	2.2	0.3	0.1	3.3	1.3	2.1	2.3	0.2	1.2	1.1	0.0
3	2.3	0.2	1.2	1.1	0.0	3.1	3.2	2.2	0.3	0.1	3.3	1.3	2.1
2	0.3	0.1	3.3	1.3	2.1	2.3	0.2	1.2	1.1	0.0	3.1	3.2	2.2
1	1.1	0.0	3.1	3.2	2.2	0.3	0.1	3.3	1.3	2.1	2.3	0.2	1.2
0	1.3	2.1	2.3	0.2	1.2	1.1	0.0	3.1	3.2	2.2	0.3	0.1	3.3
−1	3.2	2.2	0.3	0.1	3.3	1.3	2.1	2.3	0.2	1.2	1.1	0.0	3.1
−2	0.2	1.2	1.1	0.0	3.1	3.2	2.2	0.3	0.1	3.3	1.3	2.1	2.3
−3	0.1	3.3	1.3	2.1	2.3	0.2	1.2	1.1	0.0	3.1	3.2	2.2	0.3
−4	0.0	3.1	3.2	2.2	0.3	0.1	3.3	1.3	2.1	2.3	0.2	1.2	1.1
−5	2.1	2.3	0.2	1.2	1.1	0.0	3.1	3.2	2.2	0.3	0.1	3.3	1.3
−6	2.2	0.3	0.1	3.3	1.3	2.1	2.3	0.2	1.2	1.1	0.0	3.1	3.2

mod $1+4i$

A \ B	8	7	6	5	4	3	2	1	0	−1	−2	−3	−4	−5	−6	−7	−8
8	0.4	0.2	2.2	2.4	1.4	1.3	2.1	2.3	1.2	1.1	0.0	3.1	3.2	0.3	0.1	3.3	3.4
7	0.3	0.1	3.3	3.4	0.4	0.2	2.2	2.4	1.4	1.3	2.1	2.3	1.2	1.1	0.0	3.1	3.2
6	1.1	0.0	3.1	3.2	0.3	0.1	3.3	3.4	0.4	0.2	2.2	2.4	1.4	1.3	2.1	2.3	1.2
5	1.3	2.1	2.3	1.2	1.1	0.0	3.1	3.2	0.3	0.1	3.3	3.4	0.4	0.2	2.2	2.4	1.4
4	0.2	2.2	2.4	1.4	1.3	2.1	2.3	1.2	1.1	0.0	3.1	3.2	2.2	0.1	3.3	3.4	0.4
3	0.1	3.3	3.4	0.4	0.2	2.2	2.4	1.4	1.3	2.1	2.3	1.2	1.1	0.0	3.1	3.2	0.3
2	0.0	3.1	3.2	0.3	0.1	3.3	3.4	0.4	0.2	2.2	2.4	1.4	1.3	2.1	2.3	1.2	1.1
1	2.1	2.3	1.2	1.1	0.0	3.1	3.2	0.3	0.1	3.3	3.4	0.4	0.2	2.2	2.4	1.4	1.3
0	2.2	2.4	1.4	1.3	2.1	2.3	1.2	1.1	0.0	3.1	3.2	0.3	0.1	3.3	3.4	0.4	0.2
−1	3.3	3.4	0.4	0.2	2.2	2.4	1.4	1.3	2.1	2.3	1.2	1.1	0.0	3.1	3.2	0.3	0.1
−2	3.1	3.2	0.3	0.1	3.3	3.4	0.4	0.2	2.2	2.4	1.4	1.3	2.1	2.3	1.2	1.1	0.0
−3	2.3	1.2	1.1	0.0	3.1	3.2	0.3	0.1	3.3	3.4	0.4	0.2	2.2	2.4	1.4	1.3	2.1
−4	2.4	1.4	1.3	2.1	2.3	1.2	1.1	0.0	3.1	3.2	0.3	0.1	3.3	3.4	0.4	0.2	2.2
−5	3.4	0.4	0.2	2.2	2.4	1.4	1.3	2.1	2.3	1.2	1.1	0.0	3.1	3.2	0.3	0.1	3.3
−6	3.2	0.3	0.1	3.3	3.4	0.4	0.2	2.2	2.4	1.4	1.3	2.1	2.3	1.2	1.1	0.0	3.1
−7	1.2	1.1	0.0	3.1	3.2	0.3	0.1	3.3	3.4	0.4	0.2	2.2	2.4	1.4	1.3	2.1	2.3
−8	1.4	1.3	2.1	2.3	1.2	1.1	0.0	3.1	3.2	0.3	0.1	3.3	3.4	0.4	0.2	2.2	2.4

mod $1-4i$

A \ B	8	7	6	5	4	3	2	1	0	−1	−2	−3	−4	−5	−6	−7	−8
8	0.4	0.3	0.1	3.3	1.2	1.1	0.0	3.1	3.2	1.3	2.1	2.3	2.4	1.4	2.2	0.2	3.4
7	1.2	1.1	0.0	3.1	3.2	1.3	2.1	2.3	2.4	1.4	2.2	0.2	3.4	0.4	0.3	0.1	3.3
6	3.2	1.3	2.1	2.3	2.4	1.4	2.2	0.2	3.4	0.4	0.3	0.1	3.3	1.2	1.1	0.0	3.1
5	2.4	1.4	2.2	0.2	3.4	0.4	0.3	0.1	3.3	1.3	1.1	0.0	3.1	3.2	1.3	2.1	2.3
4	3.4	0.4	0.3	0.1	3.3	1.2	1.1	0.0	3.1	3.2	1.3	2.1	2.3	2.4	1.4	2.2	0.2
3	3.3	1.2	1.1	0.0	3.1	3.2	1.3	2.1	2.3	2.4	1.4	2.2	0.2	3.4	0.4	0.3	0.1
2	3.1	3.2	1.3	2.1	2.3	2.4	1.4	2.2	0.2	3.4	0.4	0.3	0.1	3.3	1.2	1.1	0.0
1	2.3	2.4	1.4	2.2	0.2	3.4	0.4	0.3	0.1	3.3	1.2	1.1	0.0	3.1	3.2	1.3	2.1
0	0.2	3.4	0.4	0.3	0.1	3.3	1.2	1.1	0.0	3.1	3.2	1.3	2.1	2.3	2.4	1.4	2.2
−1	0.1	3.3	1.2	1.1	0.0	3.1	3.2	1.3	2.1	2.3	2.4	1.4	2.2	0.2	3.4	0.4	0.3
−2	0.0	3.1	3.2	1.3	2.1	2.3	2.4	1.4	2.2	0.2	3.4	0.4	0.3	0.1	3.3	1.2	1.1
−3	2.1	2.3	2.4	1.4	2.2	0.2	3.4	0.4	0.3	0.1	3.3	1.2	1.1	0.0	3.1	3.2	1.3
−4	2.2	0.2	3.4	0.4	0.3	0.1	3.3	1.2	1.1	0.0	3.1	3.2	1.3	2.1	2.3	2.4	1.4
−5	0.3	0.1	3.3	1.2	1.1	0.0	3.1	3.2	1.3	2.1	2.3	2.4	1.4	2.2	0.2	3.4	0.4
−6	1.1	0.0	3.1	3.2	1.3	2.1	2.3	2.4	1.4	2.2	0.2	3.4	0.4	0.3	0.1	3.3	1.2
−7	1.3	2.1	2.3	2.4	1.4	2.2	0.2	3.4	0.4	0.3	0.1	3.3	1.2	1.1	0.0	3.1	3.2
−8	1.4	2.2	0.2	3.4	0.4	0.3	0.1	3.3	1.2	1.1	0.0	3.1	3.2	1.3	2.1	2.3	2.4

mod $3+4i$

A \ B	12	11	10	9	8	7	6	5	4	3	2	1	0
12	0.3	1.5	1.4	2.1	2.4	2.6	0.5	0.2	3.6	1.6	2.2	2.5	0.6
11	0.2	3.6	1.6	2.2	2.5	0.6	0.4	0.1	3.4	3.5	2.3	1.3	1.2
10	0.1	3.4	3.5	2.3	1.3	1.2	1.1	0.0	3.1	3.2	3.3	0.3	1.5
9	0.0	3.1	3.2	3.3	0.3	1.5	1.4	2.1	2.4	2.6	0.5	0.2	3.6
8	2.1	2.4	2.6	0.5	0.2	3.6	1.6	2.2	2.5	0.6	0.4	0.1	3.4
7	2.2	2.5	0.6	0.4	0.1	3.4	3.5	2.3	1.3	1.2	1.1	0.0	3.1
6	2.3	1.3	1.2	1.1	0.0	3.1	3.2	3.3	0.3	1.5	1.4	2.1	2.4
5	3.3	0.3	1.5	1.4	2.1	2.4	2.6	0.5	0.2	3.6	1.6	2.2	2.5
4	0.5	0.2	3.6	1.6	2.2	2.5	0.6	0.4	0.1	3.4	3.5	2.3	1.3
3	0.4	0.1	3.4	3.5	2.3	1.3	1.2	1.1	0.0	3.1	3.2	3.3	0.3
2	1.1	0.0	3.1	3.2	3.3	0.3	1.5	1.4	2.1	2.4	2.6	0.5	0.2
1	1.4	2.1	2.4	2.6	0.5	0.2	3.6	1.6	2.2	2.5	0.6	0.4	0.1
0	1.6	2.2	2.5	0.6	0.4	0.1	3.4	3.5	2.3	1.3	1.2	1.1	0.0
−1	3.5	2.3	1.3	1.2	1.1	0.0	3.1	3.2	3.3	0.3	1.5	1.4	2.1
−2	3.2	3.3	0.3	1.5	1.4	2.1	2.4	2.6	0.5	0.2	3.6	1.6	2.2
−3	2.6	0.5	0.2	3.6	1.6	2.2	2.5	0.6	0.4	0.1	3.4	3.5	2.3
−4	0.6	0.4	0.1	3.4	3.5	2.3	1.3	1.2	1.1	0.0	3.1	3.2	3.3
−5	1.2	1.1	0.0	3.1	3.2	3.3	0.3	1.5	1.4	2.1	2.4	2.6	0.5
−6	1.5	1.4	2.1	2.4	2.6	0.5	0.2	3.6	1.6	2.2	2.5	0.6	0.4
−7	3.6	1.6	2.2	2.5	0.6	0.4	0.1	3.4	3.5	2.3	1.3	1.2	1.1
−8	3.4	3.5	2.3	1.3	1.2	1.1	0.0	3.1	3.2	3.3	0.3	1.5	1.4
−9	3.1	3.2	3.3	0.3	1.5	1.4	2.1	2.4	2.6	0.5	0.2	3.6	1.6
−10	2.4	2.6	0.5	0.2	3.6	1.6	2.2	2.5	0.6	0.4	0.1	3.4	3.5
−11	2.5	0.6	0.4	0.1	3.4	3.5	2.3	1.3	1.2	1.1	0.0	3.1	3.2
−12	1.3	1.2	1.1	0.0	3.1	3.2	3.3	0.3	1.5	1.4	2.1	2.4	2.6

ontinuation of table

A \ B	−1	−2	−3	−4	−5	−6	−7	−8	−9	−10	−11	−12
12	0.4	0.1	3.4	3.5	2.3	1.3	1.2	1.1	0.0	3.1	3.2	3.3
11	1.1	0.0	3.1	3.2	3.3	0.3	1.5	1.4	2.1	2.4	2.6	0.5
10	1.4	2.1	2.4	2.6	0.5	0.2	3.6	1.6	2.2	2.5	0.6	0.4
9	1.6	2.2	2.5	0.6	0.4	0.1	3.4	3.5	2.3	1.3	1.2	1.1
8	3.5	2.3	1.3	1.2	1.1	0.0	3.1	3.2	3.3	0.3	1.5	1.4
7	3.2	3.3	0.3	1.5	1.4	2.1	2.4	2.6	0.5	0.2	3.6	1.6
6	2.6	0.5	0.2	3.6	1.6	2.2	2.5	0.6	0.4	0.1	3.4	3.5
5	0.6	0.4	0.1	3.4	3.5	2.3	1.3	1.2	1.1	0.0	3.1	3.2
4	1.2	1.1	0.0	3.1	3.2	3.3	0.3	1.5	1.4	2.1	2.4	2.6
3	1.5	1.4	2.1	2.4	2.6	0.5	0.2	3.6	1.6	2.2	2.5	0.6
2	3.6	1.6	2.2	2.5	0.6	0.4	0.1	3.4	3.5	2.3	1.3	1.2
1	3.4	3.5	2.3	1.3	1.2	1.1	0.0	3.1	3.2	3.3	0.3	1.5
0	3.1	3.2	3.3	0.3	1.5	1.4	2.1	2.4	2.6	0.5	0.2	3.6
−1	2.4	2.6	0.5	0.2	3.6	1.6	2.2	2.5	0.6	0.4	0.1	3.4
−2	2.5	0.6	0.4	0.1	3.4	3.5	2.3	1.3	1.2	1.1	0.0	3.1
−3	1.3	1.2	1.1	0.0	3.1	3.2	3.3	0.3	1.5	1.4	2.1	2.4
−4	0.3	1.5	1.4	2.1	2.4	2.6	0.5	0.2	3.6	1.6	2.2	2.5
−5	0.2	3.6	1.6	2.2	2.5	0.6	0.4	0.1	3.4	3.5	2.3	1.3
−6	0.1	3.4	3.5	2.3	1.3	1.2	1.1	0.0	3.1	3.2	3.3	0.3
−7	0.0	3.1	3.2	3.3	0.3	1.5	1.4	2.1	2.4	2.6	0.5	0.2
−8	2.1	2.4	2.6	0.5	0.2	3.6	1.6	2.2	2.5	0.6	0.4	0.1
−9	2.2	2.5	0.6	0.4	0.1	3.4	3.5	2.3	1.3	1.2	1.1	0.0
−10	2.3	1.3	1.2	1.1	0.0	3.1	3.2	3.3	0.3	1.5	1.4	2.1
−11	3.3	0.3	1.5	1.4	2.1	2.4	2.6	0.5	0.2	3.6	1.6	2.2
−12	0.5	0.2	3.6	1.6	2.2	2.5	0.6	0.4	0.1	3.4	3.5	2.3

mod 3—4t

A \ B	12	11	10	9	8	7	6	5	4	3	2	1	0
12	1.3	1.2	1.1	0.0	3.1	3.2	3.3	2.3	0.6	0.4	0.1	3.4	3.5
11	3.6	1.5	1.4	2.1	1.4	2.6	0.3	1.3	1.2	1.1	0.0	3.1	3.2
10	3.4	3.5	1.6	2.2	2.5	0.5	0.2	3.6	1.5	1.4	2.1	2.4	2.6
9	3.1	3.2	3.3	2.3	0.6	0.4	0.1	3.4	3.5	1.6	2.2	2.5	0.5
8	2.4	2.6	0.3	1.3	1.2	1.1	0.0	3.1	3.2	3.3	2.3	0.6	0.4
7	2.5	0.5	0.2	3.6	1.5	1.4	2.1	2.4	2.6	0.3	1.3	1.2	1.1
6	0.6	0.4	0.1	3.4	3.5	1.6	2.2	2.5	0.5	0.2	3.6	1.5	1.4
5	1.2	1.1	0.0	3.1	3.2	3.3	2.3	0.6	0.4	0.1	3.4	3.5	1.6
4	1.5	1.4	2.1	2.4	2.6	0.3	1.3	1.2	1.1	0.0	3.1	3.2	3.3
3	3.5	1.6	2.2	2.5	0.5	0.2	3.6	1.5	1.4	2.1	2.4	2.6	0.3
2	3.2	3.3	2.3	0.6	0.4	0.1	3.4	3.5	1.6	2.2	2.5	0.5	0.2
1	2.6	0.3	1.3	1.2	1.1	0.0	3.1	3.2	3.3	2.3	0.6	0.4	0.1
0	0.5	0.2	3.6	1.3	1.4	2.1	2.4	2.6	0.3	1.3	1.2	1.1	0.0
−1	0.4	0.1	3.4	3.5	1.6	2.2	2.5	0.5	0.2	3.6	1.5	1.4	2.1
−2	1.1	0.0	3.1	3.2	3.3	2.3	0.6	0.4	0.1	3.4	3.5	1.6	2.2
−3	1.4	2.1	2.4	2.6	0.3	1.3	1.2	1.1	0.0	3.1	3.2	3.3	2.3
−4	1.6	2.2	2.5	0.5	0.2	3.6	1.5	1.4	2.1	2.4	2.6	0.3	1.3
−5	3.3	2.3	0.6	0.4	0.1	3.4	3.5	1.6	2.2	2.5	0.5	0.2	3.6
−6	0.3	1.3	1.2	1.1	0.0	3.1	3.2	3.3	2.3	0.6	0.4	0.1	3.4
−7	0.2	3.6	1.5	1.4	2.1	2.4	2.6	0.3	1.3	1.2	1.1	0.0	3.1
−8	0.1	3.4	3.5	1.6	2.2	2.5	0.5	0.2	3.6	1.5	1.4	2.1	2.4
−9	0.0	3.1	3.2	3.3	2.3	0.6	0.4	0.1	3.4	3.5	1.6	2.2	2.5
−10	2.1	2.4	2.6	0.3	1.3	1.2	1.1	0.0	3.1	3.2	3.3	2.3	0.6
−11	2.2	2.5	0.5	0.2	3.6	1.5	1.4	2.1	2.4	2.6	0.3	1.3	1.2
−12	2.3	0.6	0.4	0.1	3.4	3.5	1.6	2.2	2.5	0.5	0.2	3.6	1.5

Continuation of table.

mod $3-4i$

A \ B	−1	−2	−3	−4	−5	−6	−7	−8	−9	−10	−11	−12
12	1.6	2.2	2.5	0.5	0.2	3.6	1.5	1.4	2.1	2.4	2.6	0.3
11	3.3	2.3	0.6	0.4	0.1	3.4	3.5	1.6	2.2	2.5	0.5	0.2
10	0.3	1.3	1.2	1.1	0.0	3.1	3.2	3.3	2.3	0.6	0.4	0.1
9	0.2	3.6	1.5	1.4	2.1	2.4	2.6	0.3	1.3	1.2	1.1	0.0
8	0.1	3.4	3.5	1.6	2.2	2.5	0.5	0.2	3.6	1.5	1.4	2.1
7	0.0	3.1	3.2	3.3	2.3	0.6	0.4	0.1	3.4	3.5	1.6	2.2
6	2.1	2.4	2.6	0.3	1.3	1.2	1.1	0.0	3.1	3.2	3.3	2.3
5	2.2	2.5	0.5	0.2	3.6	1.5	1.4	2.1	2.4	2.6	0.3	1.3
4	2.3	0.6	0.4	0.1	3.4	3.5	1.6	2.2	2.5	0.5	0.2	3.6
3	1.3	1.2	1.1	0.0	3.1	3.2	3.3	2.3	0.6	0.4	0.1	3.4
2	3.6	1.5	1.4	2.1	2.4	2.6	0.3	1.3	1.2	1.1	0.0	3.1
1	3.4	3.5	1.6	2.2	2.5	0.5	0.2	3.6	1.5	1.4	2.1	2.4
0	3.1	3.2	3.3	2.3	0.6	0.4	0.1	3.4	3.5	1.6	2.2	2.5
−1	2.4	2.6	0.3	1.3	1.2	1.1	0.0	3.1	3.2	3.3	2.3	0.6
−2	2.5	0.5	0.2	3.6	1.5	1.4	2.1	2.4	2.6	0.3	1.3	1.2
−3	0.6	0.4	0.1	3.4	3.5	1.6	2.2	2.5	0.5	0.2	3.6	1.5
−4	1.2	1.1	0.0	3.1	3.2	3.3	2.3	0.6	0.4	0.1	3.4	3.5
−5	1.5	1.4	2.1	2.4	2.6	0.3	1.3	1.2	1.1	0.0	3.1	3.2
−6	3.5	1.6	2.2	2.5	0.5	0.2	3.6	1.5	1.4	2.1	2.4	2.6
−7	3.2	3.3	2.3	0.6	0.4	0.1	3.4	3.5	1.6	2.2	2.5	0.5
−8	2.6	0.3	1.3	1.2	1.1	0.0	3.1	3.2	3.3	2.3	0.6	0.4
−9	0.5	0.2	3.6	1.5	1.4	2.1	2.4	2.6	0.3	1.3	1.2	1.1
−10	0.4	0.1	3.4	3.5	1.6	2.2	2.5	0.5	0.2	3.6	1.5	1.4
−11	1.1	0.0	3.1	3.2	3.3	2.3	0.6	0.4	0.1	3.4	3.5	1.6
−12	1.4	2.1	2.4	2.6	0.3	1.3	1.2	1.1	0.0	3.1	3.2	3.3

mod $2+5i$

A \ B	14	13	12	11	10	9	8	7	6	5	4	3	2	1	0
14	0.5	1.7	1.6	2.2	2.4	2.7	1.2	1.1	0.0	3.1	3.2	0.7	0.4	0.2	3.6
13	0.4	0.2	3.6	3.7	2.5	1.5	1.4	1.3	2.1	2.3	2.6	0.6	0.3	0.1	3.3
12	0.3	0.1	3.3	3.4	3.5	0.5	1.7	1.6	2.2	2.4	2.7	1.2	1.1	0.0	3.1
11	1.1	0.0	3.1	3.2	0.7	0.4	0.2	3.6	3.7	2.5	1.5	1.4	1.3	2.1	2.3
10	1.3	2.1	2.3	2.6	0.6	0.3	0.1	3.3	3.4	3.5	0.5	1.7	1.6	2.2	2.4
9	1.6	2.2	2.4	2.7	1.2	1.1	0.0	3.1	3.2	0.7	0.4	0.2	3.6	3.7	2.5
8	3.6	3.7	2.5	1.5	1.4	1.3	2.1	2.3	2.6	0.6	0.3	0.1	3.3	3.4	3.5
7	3.3	3.4	3.5	0.5	1.7	1.6	2.2	2.4	2.7	1.2	1.1	0.0	3.1	3.2	0.7
6	3.1	3.2	0.7	0.4	0.2	3.6	3.7	2.5	1.5	1.4	1.3	2.1	2.3	2.6	0.6
5	2.3	2.6	0.6	0.3	0.1	3.3	3.4	3.5	0.5	1.7	1.6	2.2	2.4	2.7	1.2
4	2.4	2.7	1.2	1.1	0.0	3.1	3.2	0.7	0.4	0.2	3.6	3.7	2.5	1.5	1.4
3	2.5	1.5	1.4	1.3	2.1	2.3	2.6	0.6	0.3	0.1	3.3	3.4	3.5	0.5	1.7
2	3.5	0.5	1.7	1.6	2.2	2.4	2.7	1.2	1.1	0.0	3.1	3.2	0.7	0.4	0.2
1	0.7	0.4	0.2	3.6	3.7	2.5	1.5	1.4	1.3	2.1	2.3	2.6	0.6	0.3	0.1
0	0.6	0.3	0.1	3.3	3.4	3.5	0.5	1.7	1.6	2.2	2.4	2.7	1.2	1.1	0.0
−1	1.2	1.1	0.0	3.1	3.2	0.7	0.4	0.2	3.6	3.7	2.5	1.5	1.4	1.3	2.1
−2	1.4	1.3	2.1	2.3	2.6	0.6	0.3	0.1	3.3	3.4	3.5	0.5	1.7	1.6	2.2
−3	1.7	1.6	2.2	2.4	2.7	1.2	1.1	0.0	3.1	3.2	0.7	0.4	0.2	3.6	3.7
−4	0.2	3.6	3.7	2.5	1.5	1.4	1.3	2.1	2.3	2.6	0.6	0.3	0.1	3.3	3.4
−5	0.1	3.3	3.4	3.5	0.5	1.7	1.6	2.2	2.4	2.7	1.2	1.1	0.0	3.1	3.2
−6	0.0	3.1	3.2	0.7	0.4	0.2	3.6	3.7	2.5	1.5	1.4	1.3	2.1	2.3	2.6
−7	2.1	2.3	2.6	0.6	0.3	0.1	3.3	3.4	3.5	0.5	1.7	1.6	2.2	2.4	2.7
−8	2.2	2.4	2.7	1.2	1.1	0.0	3.1	3.2	0.7	0.4	0.2	3.6	3.7	2.5	1.5
−9	3.7	2.5	1.5	1.4	1.3	2.1	2.3	2.6	0.6	0.3	0.1	3.3	3.4	3.5	0.5
−10	3.4	3.5	0.5	1.7	1.6	2.2	2.4	2.7	1.2	1.1	0.0	3.1	3.2	0.7	0.4
−11	3.2	0.7	0.4	0.2	3.6	3.7	2.5	1.5	1.4	1.3	2.1	2.3	2.6	0.6	0.3
−12	2.6	0.6	0.3	0.1	3.3	3.4	3.5	0.5	1.7	1.6	2.2	2.4	2.7	1.2	1.1
−13	2.7	1.2	1.1	0.0	3.1	3.2	0.7	0.4	0.2	3.6	3.7	2.5	1.5	1.4	1.3
−14	1.5	1.4	1.3	2.1	2.3	2.6	0.6	0.3	0.1	3.3	3.4	3.5	0.5	1.7	1.6

Continuation of table.

mod $2+5i$

A \ B	−1	−2	−3	−4	−5	−6	−7	−8	−9	−10	−11	−12	−13	−14
14	3.7	2.5	1.5	1.4	1.3	2.1	2.3	2.6	0.6	0.3	0.1	3.3	3.4	3.5
13	3.4	3.5	0.5	1.7	1.6	2.2	2.4	2.7	1.2	1.1	0.0	3.1	3.2	0.7
12	3.2	0.7	0.4	0.2	3.6	3.7	2.5	1.5	1.4	1.3	2.1	2.3	2.6	0.6
11	2.6	0.6	0.3	0.1	3.3	3.4	3.5	0.5	1.7	1.6	2.2	2.4	2.7	1.2
10	2.7	1.2	1.1	0.0	3.1	3.2	0.7	0.4	0.2	3.6	3.7	2.5	1.5	1.4
9	1.5	1.4	1.3	2.1	2.3	2.6	0.6	0.3	0.1	3.3	3.4	3.5	0.5	1.7
8	0.5	1.7	1.6	2.2	2.4	2.7	1.2	1.1	0.0	3.1	3.2	0.7	0.4	0.2
7	0.4	0.2	3.6	3.7	2.5	1.5	1.4	1.3	2.1	2.3	2.6	0.6	0.3	0.1
6	0.3	0.1	3.3	3.4	3.5	0.5	1.7	1.6	2.2	2.4	2.7	1.2	1.1	0.0
5	1.1	0.0	3.1	3.2	0.7	0.4	0.2	3.6	3.7	2.5	1.5	1.4	1.3	2.1
4	1.3	2.1	2.3	2.6	0.6	0.3	0.1	3.3	3.4	3.5	0.5	1.7	1.6	2.2
3	1.6	2.2	2.4	2.7	1.2	1.1	0.0	3.1	3.2	0.7	0.4	0.2	3.6	3.7
2	3.6	3.7	2.5	1.5	1.4	1.3	2.1	2.3	2.6	0.6	0.3	0.1	3.3	3.4
1	3.3	3.4	3.5	0.5	1.7	1.6	2.2	2.4	2.7	1.2	1.1	0.0	3.1	3.2
0	3.1	3.2	0.7	0.4	0.2	3.6	3.7	2.5	1.5	1.4	1.3	2.1	2.3	2.6
−1	2.3	2.6	0.6	0.3	0.1	3.3	3.4	3.5	0.5	1.7	1.6	2.2	2.4	2.7
−2	2.4	2.7	1.2	1.1	0.0	3.1	3.2	0.7	0.4	0.2	3.6	3.7	2.5	1.5
−3	2.5	1.5	1.4	1.3	2.1	2.3	2.6	0.6	0.3	0.1	3.3	3.4	3.5	0.5
−4	3.5	0.5	1.7	1.6	2.2	2.4	2.7	1.2	1.1	0.0	3.1	3.2	0.7	0.4
−5	0.7	0.4	0.2	3.6	3.7	2.5	1.5	1.4	1.3	2.1	2.3	2.6	0.6	0.3
−6	0.6	0.3	0.1	3.3	3.4	3.5	0.5	1.7	1.6	2.2	2.4	2.7	1.2	1.1
−7	1.2	1.1	0.0	3.1	3.2	0.7	0.4	0.2	3.6	3.7	2.5	1.5	1.4	1.3
−8	1.4	1.3	2.1	2.3	2.6	0.6	0.3	0.1	3.3	3.4	3.5	0.5	1.7	1.6
−9	1.7	1.6	2.3	2.4	2.7	1.2	1.1	0.0	3.1	3.2	0.7	0.4	0.2	3.6
−10	0.2	3.6	3.7	2.5	1.5	1.4	1.3	2.1	2.3	2.6	0.6	0.3	0.1	3.3
−11	0.1	3.3	3.4	3.5	0.5	1.7	1.6	2.2	2.4	2.7	1.2	1.1	0.0	3.1
−12	0.0	3.1	3.2	0.7	0.4	0.2	3.6	3.7	2.5	1.5	1.4	1.3	2.1	2.3
−13	2.1	2.3	2.6	0.6	0.3	0.1	3.3	3.4	3.5	0.5	1.7	1.6	2.2	2.4
−14	2.2	2.4	2.7	1.2	1.1	0.0	3.1	3.2	0.7	0.4	0.2	3.6	3.7	2.5

mod $2-5i$

A \ B	14	13	12	11	10	9	8	7	6	5	4	3	2	1	0
14	0.7	0.5	0.3	0.1	3.3	3.4	1.4	1.3	2.1	2.3	2.5	2.7	1.7	0.6	0.4
13	3.6	1.2	1.1	0.0	3.1	3.2	1.6	1.5	2.2	2.4	2.6	3.7	0.7	0.5	0.3
12	3.4	1.4	1.3	2.1	2.3	2.5	2.7	1.7	0.6	0.4	0.2	3.5	3.6	1.2	1.1
11	3.2	1.6	1.5	2.2	2.4	2.6	3.7	0.7	0.5	0.3	0.1	3.3	3.4	1.4	1.3
10	2.5	2.7	1.7	0.6	0.4	0.2	3.5	3.6	1.2	1.1	0.0	3.1	3.2	1.6	1.5
9	2.6	3.7	0.7	0.5	0.3	0.1	3.3	3.4	1.4	1.3	2.1	2.3	2.5	2.7	1.7
8	0.2	3.5	3.6	1.2	1.1	0.0	3.1	3.2	1.6	1.5	2.2	2.4	2.6	3.7	0.7
7	0.1	3.3	3.4	1.4	1.3	2.1	2.3	2.5	2.7	1.7	0.6	0.4	0.2	3.5	3.6
6	0.0	3.1	3.2	1.6	1.5	2.2	2.4	2.6	3.7	0.7	0.5	0.3	0.1	3.3	3.4
5	2.1	2.3	2.5	2.7	1.7	0.6	0.4	0.2	3.5	3.6	1.2	1.1	0.0	3.1	3.2
4	2.2	2.4	2.6	3.7	0.7	0.5	0.3	0.1	3.3	3.4	1.4	1.3	2.1	2.3	2.5
3	0.6	0.4	0.2	3.5	3.6	1.2	1.1	0.0	3.1	3.2	1.6	1.5	2.2	2.4	2.6
2	0.5	0.3	0.1	3.3	3.4	1.4	1.3	2.1	2.3	2.5	2.7	1.7	0.6	0.4	0.2
1	1.2	1.1	0.0	3.1	3.2	1.6	1.5	2.2	2.4	2.6	3.7	0.7	0.5	0.3	0.1
0	1.4	1.3	2.1	2.3	2.5	2.7	1.7	0.6	0.4	0.2	3.5	3.6	1.2	1.1	0.0
−1	1.6	1.5	2.2	2.4	2.6	3.7	0.7	0.5	0.3	0.1	3.3	3.4	1.4	1.3	2.1
−2	2.7	1.7	0.6	0.4	0.2	3.5	3.6	1.2	1.1	0.0	3.1	3.2	1.6	1.5	2.2
−3	3.7	0.7	0.5	0.3	0.1	3.3	3.4	1.4	1.3	2.1	2.3	2.5	2.7	1.7	0.6
−4	3.5	3.6	1.2	1.1	0.0	3.1	3.2	1.6	1.5	2.2	2.4	2.6	3.7	0.7	0.5
−5	3.3	3.4	1.4	1.3	2.1	2.3	2.5	2.7	1.7	0.6	0.4	0.2	3.5	3.6	1.2
−6	3.1	3.2	1.6	1.5	2.2	2.4	2.6	3.7	0.7	0.5	0.3	0.1	3.3	3.4	1.4
−7	2.3	2.5	2.7	1.7	0.6	0.4	0.2	3.5	3.6	1.2	1.1	0.0	3.1	3.2	1.6
−8	2.4	2.6	3.7	0.7	0.5	0.3	0.1	3.3	3.4	1.4	1.3	2.1	2.3	2.5	2.7
−9	0.4	0.2	3.5	3.6	1.2	1.1	0.0	3.1	3.2	1.6	1.5	2.2	2.4	2.6	3.7
−10	0.3	0.1	3.3	3.4	1.4	1.3	2.1	2.3	2.5	2.7	1.7	0.6	0.4	0.2	3.5
−11	1.1	0.0	3.1	3.2	1.6	1.5	2.2	2.4	2.6	3.7	0.7	0.5	0.3	0.1	3.3
−12	1.3	2.1	2.3	2.5	2.7	1.7	0.6	0.4	0.2	3.5	3.6	1.2	1.1	0.0	3.1
−13	1.5	2.2	2.4	2.6	3.7	0.7	0.5	0.3	0.1	3.3	3.4	1.4	1.3	2.1	2.3
−14	1.7	0.6	0.4	0.2	3.5	3.6	1.2	1.1	0.0	3.1	3.2	1.6	1.5	2.2	2.4

Continuation of table.

mod 2—5i

A \ B	−1	−2	−3	−4	−5	−6	−7	−8	−9	−10	−11	−12	−13	−14
14	0.2	3.5	3.6	1.2	1.1	0.0	3.1	3.2	1.6	1.5	2.2	2.4	2.6	3.7
13	0.1	3.3	3.4	1.4	1.3	2.1	2.3	2.5	2.7	1.7	0.6	0.4	0.2	3.5
12	0.0	3.1	3.2	1.6	1.5	2.2	2.4	2.6	3.7	0.7	0.5	0.3	0.1	3.3
11	2.1	2.3	2.5	2.7	1.7	0.6	0.4	0.2	3.5	3.6	1.2	1.1	0.0	3.1
10	2.2	2.4	2.6	3.7	0.7	0.5	0.3	0.1	3.3	3.4	1.4	1.3	2.1	2.3
9	0.6	0.4	0.2	3.5	3.6	1.2	1.1	0.0	3.1	3.2	1.6	1.5	2.2	2.4
8	0.5	0.3	0.1	3.3	3.4	1.4	1.3	2.1	2.3	2.5	2.7	1.7	0.6	0.4
7	1.2	1.1	0.0	3.1	3.2	1.6	1.5	2.2	2.4	2.6	3.7	0.7	0.5	0.3
6	1.4	1.3	2.1	2.3	2.5	2.7	1.7	0.6	0.4	0.2	3.5	3.6	1.2	1.1
5	1.6	1.5	2.2	2.4	2.6	3.7	0.7	0.5	0.3	0.1	3.3	3.4	1.4	1.3
4	2.7	1.7	0.6	0.4	0.2	3.5	3.6	1.2	1.1	0.0	3.1	3.2	1.6	1.5
3	3.7	0.7	0.5	0.3	0.1	3.3	3.4	1.4	1.3	2.1	2.3	2.5	2.7	1.7
2	3.5	3.6	1.2	1.1	0.0	3.1	3.2	1.6	1.5	2.2	2.4	2.6	3.7	0.7
1	3.3	3.4	1.4	1.3	2.1	2.3	2.5	2.7	1.7	0.6	0.4	0.2	3.5	3.6
0	3.1	3.2	1.6	1.5	2.2	2.4	2.6	3.7	0.7	0.5	0.3	0.1	3.3	3.4
−1	2.3	2.5	2.7	1.7	0.6	0.4	0.2	3.5	3.6	1.2	1.1	0.0	3.1	3.2
−2	2.4	2.6	3.7	0.7	0.5	0.3	0.1	3.3	3.4	1.4	1.3	2.1	2.3	2.5
−3	0.4	0.2	3.5	3.6	1.2	1.1	0.0	3.1	3.2	1.6	1.5	2.2	2.4	2.6
−4	0.3	0.1	3.3	3.4	1.4	1.3	2.1	2.3	2.5	2.7	1.7	0.6	0.4	0.2
−5	1.1	0.0	3.1	3.2	1.6	1.5	2.2	2.4	2.6	3.7	0.7	0.5	0.3	0.1
−6	1.3	2.1	2.3	2.5	2.7	1.7	0.6	0.4	0.2	3.5	3.6	1.2	1.1	0.0
−7	1.5	2.2	2.4	2.6	3.7	0.7	0.5	0.3	0.1	3.3	3.4	1.4	1.3	2.1
−8	1.7	0.6	0.4	0.2	3.5	3.6	1.2	1.1	0.0	3.1	3.2	1.6	1.5	2.2
−9	0.7	0.5	0.3	0.1	3.3	3.4	1.4	1.3	2.1	2.3	2.5	2.7	1.7	0.6
−10	3.6	1.2	1.1	0.0	3.1	3.2	1.6	1.5	2.2	2.4	2.6	3.7	0.7	0.5
−11	3.4	1.4	1.3	2.1	2.3	2.5	2.7	1.7	0.6	0.4	0.2	3.5	3.6	1.2
−12	3.2	1.6	1.5	2.2	2.4	2.6	3.7	0.7	0.5	0.3	0.1	3.3	3.4	1.4
−13	2.5	2.7	1.7	0.6	0.4	0.2	3.5	3.6	1.2	1.1	0.0	3.1	3.2	1.6
−14	2.6	3.7	0.7	0.5	0.3	0.1	3.3	3.4	1.4	1.3	2.1	2.3	2.5	2.7

mod 1+6i

A \ B	18	17	16	15	14	13	12	11	10	9	8	7	6	5	4	3	2	1	0
18	0.9	0.6	0.8	2.3	2.6	2.9	1.9	1.8	1.7	2.2	2.5	2.8	1.6	1.5	1.4	2.1	2.4	2.7	1.3
17	0.8	0.5	0.2	3.7	3.8	3.9	0.9	0.6	0.3	2.3	2.6	2.9	1.9	1.8	1.7	2.2	2.5	2.8	1.6
16	0.7	0.4	0.1	3.4	3.5	3.6	0.8	0.5	0.2	3.7	3.8	3.9	0.9	0.6	0.3	2.3	2.6	2.9	1.9
15	1.2	1.1	0.0	3.1	3.2	3.3	0.7	0.4	0.1	3.4	3.5	3.6	0.8	0.5	0.2	3.7	3.8	3.9	0.9
14	1.5	1.4	2.1	2.4	2.7	1.3	1.2	1.1	0.0	3.1	3.2	3.3	0.7	0.4	0.1	3.4	3.5	3.6	0.8
13	1.8	1.7	2.2	2.5	2.8	1.6	1.5	1.4	2.1	2.4	2.7	1.3	1.2	1.1	0.0	3.1	3.2	3.3	0.7
12	0.6	0.3	2.3	2.6	2.9	1.9	1.8	1.7	2.2	2.5	2.8	1.6	1.5	1.4	2.1	2.4	2.7	1.3	1.2
11	0.5	0.2	3.7	3.8	3.9	0.9	0.6	0.3	2.3	2.6	2.9	1.9	1.8	1.7	2.2	2.5	2.8	1.6	1.5
10	0.4	0.1	3.4	3.5	3.6	0.8	0.5	0.2	3.7	3.8	3.9	0.9	0.6	0.3	2.3	2.6	2.9	1.9	1.8
9	1.1	0.0	3.1	3.2	3.3	0.7	0.4	0.1	3.4	3.5	3.6	0.8	0.5	0.2	3.7	3.8	3.9	0.9	0.6
8	1.4	2.1	2.4	2.7	1.3	1.2	1.1	0.0	3.1	3.2	3.3	0.7	0.4	0.1	3.4	3.5	3.6	0.8	0.5
7	1.7	2.2	2.5	2.8	1.6	1.5	1.4	2.1	2.4	2.7	1.3	1.2	1.1	0.0	3.1	3.2	3.3	0.7	0.4
6	0.3	2.3	2.6	2.9	1.9	1.8	1.7	2.2	2.5	2.8	1.6	1.5	1.4	2.1	2.4	2.7	1.3	1.2	1.1
5	0.2	3.7	3.8	3.9	0.9	0.6	0.3	2.3	2.6	2.9	1.9	1.8	1.7	2.2	2.5	2.8	1.6	1.5	1.4
4	0.1	3.4	3.5	3.6	0.8	0.5	0.2	3.7	3.8	3.9	0.9	0.6	0.3	2.3	2.6	2.9	1.9	1.8	1.7
3	0.0	3.1	3.2	3.3	0.7	0.4	0.1	3.4	3.5	3.6	0.8	0.5	0.2	3.7	3.8	3.9	0.9	0.6	0.3
2	2.1	2.4	2.7	1.3	1.2	1.1	0.0	3.1	3.2	3.3	0.7	0.4	0.1	3.4	3.5	3.6	0.8	0.5	0.2
1	2.2	2.5	2.8	1.6	1.5	1.4	2.1	2.4	2.7	1.3	1.2	1.1	0.0	3.1	3.2	3.3	0.7	0.4	0.1
0	2.3	2.6	2.9	1.9	1.8	1.7	2.2	2.5	2.8	1.6	1.5	1.4	2.1	2.4	2.7	1.3	1.2	1.1	0.0
−1	3.7	3.8	3.9	0.9	0.6	0.3	2.3	2.6	2.9	1.9	1.8	1.7	2.2	2.5	2.8	1.6	1.5	1.4	2.1
−2	3.4	3.5	3.6	0.8	0.5	0.2	3.7	3.8	3.9	0.9	0.6	0.3	2.3	2.6	2.9	1.9	1.8	1.7	2.2
−3	3.1	3.2	3.3	0.7	0.4	0.1	3.4	3.5	3.6	0.8	0.5	0.2	3.7	3.8	3.9	0.9	0.6	0.3	2.3
−4	2.4	2.7	1.3	1.2	1.1	0.0	3.1	3.2	3.3	0.7	0.4	0.1	3.4	3.5	3.6	0.8	0.5	0.2	3.7
−5	2.5	2.8	1.6	1.5	1.4	2.1	2.4	2.7	1.3	1.2	1.1	0.0	3.1	3.2	3.3	0.7	0.4	0.1	3.4
−6	2.6	2.9	1.9	1.8	1.7	2.2	2.5	2.8	1.6	1.5	1.4	2.1	2.4	2.7	1.3	1.2	1.1	0.0	3.1
−7	3.8	3.9	0.9	0.6	0.3	2.3	2.6	2.9	1.9	1.8	1.7	2.2	2.5	2.8	1.6	1.5	1.4	2.1	2.4
−8	3.5	3.6	0.8	0.5	0.2	3.7	3.8	3.9	0.9	0.6	0.3	2.3	2.6	2.9	1.9	1.8	1.7	2.2	2.5
−9	3.2	3.3	0.7	0.4	0.1	3.4	3.5	3.6	0.8	0.5	0.2	3.7	3.8	3.9	0.9	0.6	0.3	2.3	2.6
−10	2.7	1.3	1.2	1.1	0.0	3.1	3.2	3.3	0.7	0.4	0.1	3.4	3.5	3.6	0.8	0.5	0.2	3.7	3.8
−11	2.8	1.6	1.5	1.4	2.1	2.4	2.7	1.3	1.2	1.1	0.0	3.1	3.2	3.3	0.7	0.4	0.1	3.4	3.5
−12	2.9	1.9	1.8	1.7	2.2	2.5	2.8	1.6	1.5	1.4	2.1	2.4	2.7	1.3	1.2	1.1	0.0	3.1	3.2
−13	3.9	0.9	0.6	0.3	2.3	2.6	2.9	1.9	1.8	1.7	2.2	2.5	2.8	1.6	1.5	1.4	2.1	2.4	2.7
−14	3.6	0.8	0.5	0.2	3.7	3.8	3.9	0.9	0.6	0.3	2.3	2.6	2.9	1.9	1.8	1.7	2.2	2.5	2.8
−15	3.3	0.7	0.4	0.1	3.4	3.5	3.6	0.8	0.5	0.2	3.7	3.8	3.9	0.9	0.6	0.3	2.3	2.6	2.9
−16	1.3	1.2	1.1	0.0	3.1	3.2	3.3	0.7	0.4	0.1	3.4	3.5	3.6	0.8	0.5	0.2	3.7	3.8	3.9
−17	1.6	1.5	1.4	2.1	2.4	2.7	1.3	1.2	1.1	0.0	3.1	3.2	3.3	0.7	0.4	0.1	3.4	3.5	3.6
−18	1.9	1.8	1.7	2.2	2.5	2.8	1.6	1.5	1.4	2.1	2.4	2.7	1.3	1.2	1.1	0.0	3.1	3.2	3.3

Continuation of table.

mod $1+6i$

A \ B	−1	−2	−3	−4	−5	−6	−7	−8	−9	−10	−11	−12	−13	−14	−15	−16	−17	−18
18	1.2	1.1	0.0	3.1	3.2	3.3	0.7	0.4	0.1	3.4	3.5	3.6	0.8	0.5	0.2	3.7	3.8	3.9
17	1.5	1.4	2.1	2.4	2.7	1.3	1.2	1.1	0.0	3.1	3.2	3.3	0.7	0.4	0.1	3.4	3.5	3.6
16	1.8	1.7	2.2	2.5	2.8	1.6	1.5	1.4	2.1	2.4	2.7	1.3	1.2	1.1	0.0	3.1	3.2	3.3
15	0.6	0.3	2.3	2.6	2.9	1.9	1.8	1.7	2.2	2.5	2.8	1.6	1.5	1.4	2.1	2.4	2.7	1.3
14	0.5	0.2	3.7	3.8	3.9	0.9	0.6	0.3	2.3	2.6	2.9	1.9	1.8	1.7	2.2	2.5	2.8	1.6
13	0.4	0.1	3.4	3.5	3.6	0.8	0.5	0.2	3.7	3.8	3.9	0.9	0.6	0.3	2.3	2.6	2.9	1.9
12	1.1	0.0	3.1	3.2	3.3	0.7	0.4	0.1	3.4	3.5	3.6	0.8	0.5	0.2	3.7	3.8	3.9	0.9
11	1.4	2.1	2.4	2.7	1.3	1.2	1.1	0.0	3.1	3.2	3.3	0.7	0.4	0.1	3.4	3.5	3.6	0.8
10	1.7	2.2	2.5	2.8	1.6	1.5	1.4	2.1	2.4	2.7	1.3	1.2	1.1	0.0	3.1	3.2	3.3	0.7
9	0.3	2.3	2.6	2.9	1.9	1.8	1.7	2.2	2.5	2.8	1.6	1.5	1.4	2.1	2.4	2.7	1.3	1.2
8	0.2	3.7	3.8	3.9	0.9	0.6	0.3	2.3	2.6	2.9	1.9	1.8	1.7	2.2	2.5	2.8	1.6	1.5
7	0.1	3.4	3.5	3.6	0.8	0.5	0.2	3.7	3.8	3.9	0.9	0.6	0.3	2.3	2.6	2.9	1.9	1.8
6	0.0	3.1	3.2	3.3	0.7	0.4	0.1	3.4	3.5	3.6	0.8	0.5	0.2	3.7	3.8	3.9	0.9	0.6
5	2.1	2.4	2.7	1.3	1.2	1.1	0.0	3.1	3.2	3.3	0.7	0.4	0.1	3.4	3.5	3.6	0.8	0.5
4	2.2	2.5	2.8	1.6	1.5	1.4	2.1	2.4	2.7	1.3	1.2	1.1	0.0	3.1	3.2	3.3	0.7	0.4
3	2.3	2.6	2.9	1.9	1.8	1.7	2.2	2.5	2.8	1.6	1.5	1.4	2.1	2.4	2.7	1.3	1.2	1.1
2	3.7	3.8	3.9	0.9	0.6	0.3	2.3	2.6	2.9	1.9	1.8	1.7	2.2	2.5	2.8	1.6	1.5	1.4
1	3.4	3.5	3.6	0.8	0.5	0.2	3.7	3.8	3.9	0.9	0.6	0.3	2.3	2.6	2.9	1.9	1.8	1.7
0	3.1	3.2	3.3	0.7	0.4	0.1	3.4	3.5	3.6	0.8	0.5	0.2	3.7	3.8	3.9	0.9	0.6	0.3
−1	2.4	2.7	1.3	1.2	1.1	0.0	3.1	3.2	3.3	0.7	0.4	0.1	3.4	3.5	3.6	0.8	0.5	0.2
−2	2.5	2.8	1.6	1.5	1.4	2.1	2.4	2.7	1.3	1.2	1.1	0.0	3.1	3.2	3.3	0.7	0.4	0.1
−3	2.6	2.9	1.9	1.8	1.7	2.2	2.5	2.8	1.6	1.5	1.4	2.1	2.4	2.7	1.3	1.2	1.1	0.0
−4	3.8	3.9	0.9	0.6	0.3	2.3	2.6	2.9	1.9	1.8	1.7	2.2	2.5	2.8	1.6	1.5	1.4	2.1
−5	3.5	3.6	0.8	0.5	0.2	3.7	3.8	3.9	0.9	0.6	0.3	2.3	2.6	2.9	1.9	1.8	1.7	2.2
−6	3.2	3.3	0.7	0.4	0.1	3.4	3.5	3.6	0.8	0.5	0.2	3.7	3.8	3.9	0.9	0.6	0.3	2.3
−7	2.7	1.3	1.2	1.1	0.0	3.1	3.2	3.3	0.7	0.4	0.1	3.4	3.5	3.6	0.8	0.5	0.2	3.7
−8	2.8	1.6	1.5	1.4	2.1	2.4	2.7	1.3	1.2	1.1	0.0	3.1	3.2	3.3	0.7	0.4	0.1	3.4
−9	2.9	1.9	1.8	1.7	2.2	2.5	2.8	1.6	1.5	1.4	2.1	2.4	2.7	1.3	1.2	1.1	0.0	3.1
−10	3.9	0.9	0.6	0.3	2.3	2.6	2.9	1.9	1.8	1.7	2.2	2.5	2.8	1.6	1.5	1.4	2.1	2.4
−11	3.6	0.8	0.5	0.2	3.7	3.8	3.9	0.9	0.6	0.3	2.3	2.6	2.9	1.9	1.8	1.7	2.2	2.5
−12	3.3	0.7	0.4	0.1	3.4	3.5	3.6	0.8	0.5	0.2	3.7	3.8	3.9	0.9	0.6	0.3	2.3	2.6
−13	1.3	1.2	1.1	0.0	3.1	3.2	3.3	0.7	0.4	0.1	3.4	3.5	3.6	0.8	0.5	0.2	3.7	3.8
−14	1.6	1.5	1.4	2.1	2.4	2.7	1.3	1.2	1.1	0.0	3.1	3.2	3.3	0.7	0.4	0.1	3.4	3.5
−15	1.9	1.8	1.7	2.2	2.5	2.8	1.6	1.5	1.4	2.1	2.4	2.7	1.3	1.2	1.1	0.0	3.1	3.2
−16	0.9	0.6	0.3	2.3	2.6	2.9	1.9	1.8	1.7	2.2	2.5	2.8	1.6	1.5	1.4	2.1	2.4	2.7
−17	0.8	0.5	0.2	3.7	3.8	3.9	0.9	0.6	0.3	2.3	2.6	2.9	1.9	1.8	1.7	2.2	2.5	2.8
−18	0.7	0.4	0.1	3.4	3.5	3.6	0.8	0.5	0.2	3.7	3.9	3.9	0.9	0.6	0.3	[illegible]	[illegible]	[illegible]

mod 1—6t

A \ B	18	17	16	15	14	13	12	11	10	9	8	7	6	5	4	3	2	1	0
18	0.9	0.7	0.5	0.2	3.6	3.7	0.8	0.6	0.4	0.1	3.4	3.5	1.3	1.2	1.1	0.0	3.1	3.2	3.3
17	0.8	0.6	0.4	0.1	3.4	3.5	1.3	1.2	1.1	0.0	3.1	3.2	3.3	1.5	1.4	2.1	2.4	2.6	2.8
16	1.3	1.2	1.1	0.0	3.1	3.2	3.3	1.5	1.4	2.1	2.4	2.6	2.8	1.7	1.6	2.2	2.5	2.7	2.9
15	3.3	1.5	1.4	2.1	2.4	2.6	2.8	1.7	1.6	2.2	2.5	2.7	2.9	1.9	1.8	2.3	0.3	3.8	3.9
14	2.8	1.7	1.6	2.2	2.5	2.7	2.9	1.9	1.8	2.3	0.3	3.8	3.9	0.9	0.7	0.5	0.2	3.6	3.7
13	2.9	1.9	1.8	2.3	0.3	3.8	3.9	0.9	0.7	0.5	0.2	3.6	3.7	0.8	0.6	0.4	0.1	3.4	3.5
12	3.9	0.9	0.7	0.5	0.2	3.6	3.7	0.8	0.6	0.4	0.1	3.4	3.5	1.3	1.2	1.1	0.0	3.1	3.2
11	3.7	0.8	0.6	0.4	0.1	3.4	3.5	1.3	1.2	1.1	0.0	3.1	3.2	3.3	1.5	1.4	2.1	2.4	2.6
10	3.5	1.3	1.2	1.1	0.0	3.1	3.2	3.3	1.5	1.4	2.1	2.4	2.6	2.8	1.7	1.6	2.2	2.5	2.7
9	3.2	3.3	1.5	1.4	2.1	2.4	2.6	2.8	1.7	1.6	2.2	2.5	2.7	2.9	1.9	1.8	2.3	0.3	3.8
8	2.6	[illegible]	1.7	1.6	2.2	2.5	2.7	2.9	1.9	1.8	2.3	0.3	3.8	3.9	0.9	0.7	0.5	0.2	3.6
7	2.7	2.9	1.9	1.8	2.3	0.3	3.8	3.9	0.9	0.7	0.5	0.2	3.6	3.7	0.8	0.6	0.4	0.1	3.4
6	3.8	3.9	0.9	0.7	0.5	0.2	3.6	3.7	0.8	0.6	0.4	0.1	3.4	3.5	1.3	1.2	1.1	0.0	3.1
5	3.6	3.7	0.8	0.6	0.4	0.1	3.4	3.5	1.3	1.2	1.1	0.0	3.1	3.2	3.3	1.5	1.4	2.1	2.4
4	3.4	3.5	1.3	1.2	1.1	0.0	3.1	3.2	3.3	1.5	1.4	2.1	2.4	2.6	2.8	1.7	1.6	2.2	2.5
3	3.1	3.2	3.3	1.5	1.4	2.1	2.4	2.6	2.8	1.7	1.6	2.2	2.5	2.7	2.9	1.9	1.8	2.3	0.3
2	2.4	2.6	2.8	1.7	1.6	2.2	2.5	2.7	2.9	1.9	1.8	2.3	0.3	3.8	3.9	0.9	0.7	0.5	0.2
1	2.5	2.7	2.9	1.9	1.8	2.3	0.3	3.8	3.9	0.9	0.7	0.5	0.2	3.6	3.7	0.8	0.6	0.4	0.1
0	0.3	3.8	3.9	0.9	0.7	0.5	0.2	3.6	3.7	0.8	0.6	0.4	0.1	3.4	3.5	1.3	1.2	1.1	0.0
−1	0.2	3.6	3.7	0.8	0.6	0.4	0.1	3.4	3.5	1.3	1.2	1.1	0.0	3.1	3.2	3.3	1.5	1.4	2.1
−2	0.1	3.4	3.5	1.3	1.2	1.1	0.0	3.1	3.2	3.3	1.5	1.4	2.1	2.4	2.6	2.8	1.7	1.6	2.2
−3	0.0	3.1	3.2	3.3	1.5	1.4	2.1	2.4	2.6	2.8	1.7	1.6	2.2	2.5	2.7	2.9	1.9	1.8	2.3
−4	2.1	2.4	2.6	2.8	1.7	1.6	2.2	2.5	2.7	2.9	1.9	1.8	2.3	0.3	3.8	3.9	0.9	0.7	0.5
−5	2.2	2.5	2.7	2.9	1.9	1.8	2.3	0.3	3.8	3.9	0.9	0.7	0.5	0.2	3.6	3.7	0.8	0.6	0.4
−6	2.3	0.3	3.8	3.9	0.9	0.7	0.5	0.2	3.6	3.7	0.8	0.6	0.4	0.1	3.4	3.5	1.3	1.2	1.1
−7	0.5	0.2	3.6	3.7	0.8	0.6	0.4	0.1	3.4	3.5	1.3	1.2	1.1	0.0	3.1	3.2	3.3	1.5	1.4
−8	0.4	0.1	3.4	3.5	1.3	1.2	1.1	0.0	3.1	3.2	3.3	1.5	1.4	2.1	2.4	2.6	2.8	1.7	1.6
−9	1.1	0.0	3.1	3.2	3.3	1.5	1.4	2.1	2.4	2.6	2.8	1.7	1.6	2.2	2.5	2.7	2.9	1.9	1.8
−10	1.4	2.1	2.4	2.6	2.8	1.7	1.6	2.2	2.5	2.7	2.9	1.9	1.8	2.3	0.3	3.8	3.9	0.9	0.7
−11	1.6	2.2	2.5	2.7	2.9	1.9	1.8	2.3	0.3	3.8	3.9	0.9	0.7	0.5	0.2	3.6	3.7	0.8	0.6
−12	1.8	2.3	0.3	3.8	3.9	0.9	0.7	0.5	0.2	3.6	3.7	0.8	0.6	0.4	0.1	3.4	3.5	1.3	1.2
−13	0.7	0.5	0.2	3.6	3.7	0.8	0.6	0.4	0.1	3.4	3.5	1.3	1.2	1.1	0.0	3.1	3.2	3.3	1.5
−14	0.6	0.4	0.1	3.4	3.5	1.3	1.2	1.1	0.0	3.1	3.2	3.3	1.5	1.4	2.1	2.4	2.6	2.8	1.7
−15	1.2	1.1	0.0	3.1	3.2	3.3	1.5	1.4	2.1	2.4	2.6	2.8	1.7	1.6	2.2	2.5	2.7	2.9	1.9
−16	1.5	1.4	2.1	2.4	2.6	2.8	1.7	1.6	2.2	2.5	2.7	2.9	1.9	1.8	2.3	0.3	3.8	3.9	0.9
−17	1.7	1.6	2.2	2.5	2.7	2.9	1.9	1.8	2.3	0.3	3.8	3.9	0.9	0.7	0.5	0.2	3.6	3.7	0.8
−18	1.9	1.8	2.3	0.3	3.8	3.9	0.9	0.7	0.5	0.2	3.6	3.7	0.8	0.6	0.4	0.1	3.4	3.5	1.3

Continuation of table.

ontinuation of table.

mod $1-6i$

A \ B	−1	−2	−3	−4	−5	−6	−7	−8	−9	−10	−11	−12	−13	−14	−15	−16	−17	−18
18	1.5	1.4	2.1	2.4	2.6	2.8	1.7	1.6	2.2	2.5	2.7	2.9	1.9	1.8	2.3	0.3	3.8	3.9
17	1.7	1.6	2.2	2.5	2.7	2.9	1.9	1.8	2.3	0.3	3.8	3.9	0.9	0.7	0.5	0.2	3.6	3.7
16	1.9	1.8	2.3	0.3	3.8	3.9	0.9	0.7	0.5	0.2	3.6	3.7	0.8	0.6	0.4	0.1	3.4	3.5
15	0.9	0.7	0.5	0.2	3.6	3.7	0.8	0.6	0.4	0.1	3.4	3.5	1.3	1.2	1.1	0.0	3.1	3.2
14	0.8	0.6	0.4	0.1	3.4	3.5	1.3	1.2	1.1	0.0	3.1	3.2	3.3	1.5	1.4	2.1	2.4	2.6
13	1.3	1.2	1.1	0.0	3.1	3.2	3.3	1.5	1.4	2.1	2.4	2.6	2.8	1.7	1.6	2.2	2.5	2.7
12	3.3	1.5	1.4	2.1	2.4	2.6	2.8	1.7	1.6	2.2	2.5	2.7	2.9	1.9	1.8	2.3	0.3	3.8
11	2.8	1.7	1.6	2.2	2.5	2.7	2.9	1.9	1.8	2.3	0.3	3.8	3.9	0.9	0.7	0.5	0.2	3.6
10	2.9	1.9	1.8	2.3	0.3	3.8	3.9	0.9	0.7	0.5	0.2	3.6	3.7	0.8	0.6	0.4	0.1	3.4
9	3.9	0.9	0.7	0.5	0.2	3.6	3.7	0.8	0.6	0.4	0.1	3.4	3.5	1.3	1.2	1.1	0.0	3.1
8	3.7	0.8	0.6	0.4	0.1	3.4	3.5	1.3	1.2	1.1	0.0	3.1	3.2	3.3	1.5	1.4	2.1	2.4
7	3.5	1.3	1.2	1.1	0.0	3.1	3.2	3.3	1.5	1.4	2.1	2.4	2.6	2.8	1.7	1.6	2.2	2.5
6	3.2	3.3	1.5	1.4	2.1	2.4	2.6	2.8	1.7	1.6	2.2	2.5	2.7	2.9	1.9	1.8	2.3	0.3
5	2.6	2.8	1.7	1.6	2.2	2.5	2.7	2.9	1.9	1.8	2.3	0.3	3.8	3.9	0.9	0.7	0.5	0.2
4	2.7	2.9	1.9	1.8	2.3	0.3	3.8	3.9	0.9	0.7	0.5	0.2	3.6	3.7	0.8	0.6	0.4	0.1
3	3.8	3.9	0.9	0.7	0.5	0.2	3.6	3.7	0.8	0.6	0.4	0.1	3.4	3.5	1.3	1.2	1.1	0.0
2	3.6	3.7	0.8	0.6	0.4	0.1	3.4	3.5	1.3	1.2	1.1	0.0	3.1	3.2	3.3	1.5	1.4	2.1
1	3.4	3.5	1.3	1.2	1.1	0.0	3.1	3.2	2.3	1.5	1.4	2.1	2.4	2.6	2.8	1.7	1.6	2.2
0	3.1	3.2	3.3	1.5	1.4	2.1	2.4	2.6	3.8	1.7	1.6	2.2	2.5	2.7	2.9	1.9	1.8	2.3
−1	2.4	2.6	2.8	1.7	1.6	2.2	2.5	2.7	2.9	1.9	1.8	2.3	0.3	3.8	3.9	0.9	0.7	0.5
−2	2.5	2.7	2.9	1.9	1.8	2.3	0.3	3.8	3.9	0.9	0.7	0.5	0.2	3.6	3.7	0.8	0.6	0.4
−3	0.3	3.8	3.9	0.9	0.7	0.5	0.2	3.6	3.7	0.8	0.6	0.4	0.1	3.4	3.5	1.3	1.2	1.1
−4	0.2	3.6	3.7	0.8	0.6	0.4	0.1	3.4	3.5	1.3	1.2	1.1	0.0	3.1	3.2	3.3	1.5	1.4
−5	0.1	3.4	3.5	1.3	1.2	1.1	0.0	3.1	3.2	3.3	1.5	1.4	2.1	2.4	2.6	2.8	1.7	1.6
−6	0.0	3.1	3.2	3.3	1.5	1.4	2.1	2.4	2.6	2.8	1.7	1.6	2.2	2.5	2.7	2.9	1.9	1.8
−7	2.1	2.4	2.6	2.8	1.7	1.6	2.2	2.5	2.7	2.9	1.9	1.8	2.3	0.3	3.8	3.9	0.9	0.7
−8	2.2	2.5	2.7	2.9	1.9	1.8	2.3	0.3	3.8	3.9	0.9	0.7	0.5	0.2	3.6	3.7	0.8	0.6
−9	2.3	0.3	3.8	3.9	0.9	0.7	0.5	0.2	3.6	3.7	0.8	0.6	0.4	0.1	3.4	3.5	1.3	1.2
−10	0.5	0.2	3.6	3.7	0.8	0.6	0.4	0.1	3.4	3.5	1.3	1.2	1.1	0.0	3.1	3.2	3.3	1.5
−11	0.4	0.1	3.4	3.5	1.3	1.2	1.1	0.0	3.1	3.2	3.3	1.5	1.4	2.1	2.4	2.6	2.8	1.7
−12	1.1	0.0	3.1	3.2	3.3	1.5	1.4	2.1	2.4	2.6	2.8	1.7	1.6	2.2	2.5	2.7	2.9	1.9
−13	1.4	2.1	2.4	2.6	2.8	1.7	1.6	2.2	2.5	2.7	2.9	1.9	1.8	2.3	0.3	3.8	3.9	0.9
−14	1.6	2.2	2.5	2.7	2.9	1.9	1.8	2.3	0.3	3.8	3.9	0.9	0.7	0.5	0.2	3.6	3.7	0.8
−15	1.8	2.3	0.3	3.8	3.9	0.9	0.7	0.5	0.2	3.6	3.7	0.8	0.6	0.4	0.1	3.4	3.5	1.3
−16	0.7	0.5	0.2	3.6	3.7	0.8	0.6	0.4	0.1	3.4	3.5	1.3	1.2	1.1	0.0	3.1	3.2	3.3
−17	0.6	0.4	0.1	3.4	3.5	1.3	1.2	1.1	0.0	3.1	3.2	3.3	1.5	1.4	2.1	2.4	2.6	2.8
−18	1.2	1.1	0.0	3.1	3.2	3.3	1.5	1.4	2.1	2.4	2.6	2.8	1.7	1.6	2.2	2.5	2.7	2.9

mod $4+5i$

A \ B	20	19	18	17	16	15	14	13	12	11	10	9	8	7	6	5	4	3	2	1	0
20	0.4	1.7	1.6	1.5	2.1	2.5	2.8	2.10	0.7	0.3	3.10	1.9	1.8	2.2	2.6	2.9	0.9	0.6	0.2	3.8	3.9
19	0.3	3.10	1.9	1.8	2.2	2.6	2.9	0.9	0.6	0.2	3.8	3.9	1.10	2.3	2.7	0.10	0.8	0.5	0.1	3.5	3.6
18	0.2	3.8	3.9	1.10	2.3	2.7	0.10	0.8	0.5	0.1	3.5	3.6	3.7	2.4	1.4	1.3	1.2	1.1	0.0	3.1	3.2
17	0.1	3.5	3.6	3.7	2.4	1.4	1.3	1.2	1.1	0.0	3.1	3.2	3.3	3.4	0.4	1.7	1.6	1.5	2.1	2.5	2.8
16	0.0	3.1	3.2	3.3	3.4	0.4	1.7	1.6	1.5	2.1	2.5	2.8	2.10	0.7	0.3	3.10	1.9	1.8	2.2	2.6	2.9
15	2.1	2.5	2.8	2.10	0.7	0.3	3.10	1.9	1.8	2.2	2.6	2.9	0.9	0.6	0.2	3.8	3.9	1.10	2.3	2.7	0.10
14	2.2	2.6	2.9	0.9	0.6	0.2	3.8	3.9	1.10	2.3	2.7	0.10	0.8	0.5	0.1	3.5	3.6	3.7	2.4	1.4	1.3
13	2.3	2.7	0.10	0.8	0.5	0.1	3.5	3.6	3.7	2.4	1.4	1.3	1.2	1.1	0.0	3.1	3.2	3.3	3.4	0.4	1.7
12	2.4	1.4	1.3	1.2	1.1	0.0	3.1	3.2	3.3	3.4	0.4	1.7	1.6	1.5	2.1	2.5	2.8	2.10	0.7	0.3	3.10
11	3.4	0.4	1.7	1.6	1.5	2.1	2.5	2.8	2.10	0.7	0.3	3.10	1.9	1.8	2.2	2.6	2.9	0.9	0.6	0.2	3.8
10	0.7	0.3	3.10	1.9	1.8	2.2	2.6	2.9	0.9	0.6	0.2	3.8	3.9	1.10	2.3	2.7	0.10	0.8	0.5	0.1	3.5
9	0.6	0.2	3.8	3.9	1.10	2.3	2.7	0.10	0.8	0.5	0.1	3.5	3.6	3.7	2.4	1.4	1.3	1.2	1.1	0.0	3.1
8	0.5	0.1	3.5	3.6	3.7	2.4	1.4	1.3	1.2	1.1	0.0	3.1	3.2	3.3	3.4	0.4	1.7	1.6	1.5	2.1	2.5
7	1.1	0.0	3.1	3.2	3.3	3.4	0.4	1.7	1.6	1.5	2.1	2.5	2.8	2.10	0.7	0.3	3.10	1.9	1.8	2.2	2.6
6	1.5	2.1	2.5	2.8	2.10	0.7	0.3	3.10	1.9	1.8	2.2	2.6	2.9	0.9	0.6	0.2	3.8	3.9	1.10	2.3	2.7
5	1.8	2.2	2.6	2.9	0.9	0.6	0.2	3.8	3.9	1.10	2.3	2.7	0.10	0.8	0.5	0.1	3.5	3.6	3.7	2.4	1.4
4	1.10	2.3	2.7	0.10	0.8	0.5	0.1	3.5	3.6	3.7	2.4	1.4	1.3	1.2	1.1	0.0	3.1	3.2	3.3	3.4	0.4
3	3.7	2.4	1.4	1.3	1.2	1.1	0.0	3.1	3.2	3.3	3.4	0.4	1.7	1.6	1.5	2.1	2.5	2.8	2.10	0.7	0.3
2	3.3	3.4	0.4	1.7	1.6	1.5	2.1	2.5	2.8	2.10	0.7	0.3	3.10	1.9	1.8	2.2	2.6	2.9	0.9	0.6	0.2
1	2.10	0.7	0.3	3.10	1.9	1.8	2.2	2.6	2.9	0.9	0.6	0.2	3.8	3.9	1.10	2.3	2.7	0.10	0.8	0.5	0.1
0	0.9	0.6	0.2	3.8	3.9	1.10	2.3	2.7	0.10	0.8	0.5	0.1	3.5	3.6	3.7	2.4	1.4	1.3	1.2	1.1	0.0
−1	0.8	0.5	0.1	3.5	3.6	3.7	2.4	1.4	1.3	1.2	1.1	0.0	3.1	3.2	3.3	3.4	0.4	1.7	1.6	1.5	2.1
−2	1.2	1.1	0.0	3.1	3.2	3.3	3.4	0.4	1.7	1.6	1.5	2.1	2.5	2.8	2.10	0.7	0.3	3.10	1.9	1.8	2.2
−3	1.6	1.5	2.1	2.5	2.8	2.10	0.7	0.3	3.10	1.9	1.8	2.2	2.6	2.9	0.9	0.6	0.2	3.8	3.9	1.10	2.3
−4	1.9	1.8	2.2	2.6	2.9	0.9	0.6	0.2	3.8	3.9	1.10	2.3	2.7	0.10	0.8	0.5	0.1	3.5	3.6	3.7	2.4
−5	3.9	1.10	2.3	2.7	0.10	0.8	0.5	0.1	3.5	3.6	3.7	2.4	1.4	1.3	1.2	1.1	0.0	3.1	3.2	3.3	3.4
−6	3.6	3.7	2.4	1.4	1.3	1.2	1.1	0.0	3.1	3.2	3.3	3.4	0.4	1.7	1.6	1.5	2.1	2.5	2.8	2.10	0.7
−7	3.2	3.3	3.4	0.4	1.7	1.6	1.5	2.1	2.5	2.8	2.10	0.7	0.3	3.10	1.9	1.8	2.2	2.6	2.9	0.9	0.6
−8	2.8	2.10	0.7	0.3	3.10	1.9	1.8	2.2	2.6	2.9	0.9	0.6	0.2	3.8	3.9	1.10	2.3	2.7	0.10	0.8	0.5

Continuation of table.

mod $4+5i$

A \ B	20	19	18	17	16	15	14	13	12	11	10	9	8	7	6	5	4	3	2	1	0
−9	2.9	0.9	0.6	0.2	3.8	3.9	1.10	2.3	2.7	0.10	0.8	0.5	0.1	3.5	3.6	3.7	2.4	1.4	1.3	1.2	1.1
−10	0.10	0.8	0.5	0.1	3.5	3.6	3.7	2.4	1.4	1.3	1.2	1.1	0.0	3.1	3.2	3.3	3.4	0.4	1.7	1.6	1.5
−11	1.3	1.2	1.1	0.0	3.1	3.2	3.3	3.4	0.4	1.7	1.6	1.5	2.1	2.5	2.8	2.10	0.7	0.3	3.10	1.9	1.8
−12	1.7	1.6	1.5	2.1	2.5	2.8	2.10	0.7	0.3	3.10	1.9	1.8	2.2	2.6	2.9	0.9	0.6	0.2	3.8	3.9	1.10
−13	3.10	1.9	1.8	2.2	2.6	2.9	0.9	0.6	0.2	3.8	3.9	1.10	2.3	2.7	0.10	0.8	0.5	0.1	3.5	3.6	3.7
−14	3.8	3.9	1.10	2.3	2.7	0.10	0.8	0.5	0.1	3.5	3.6	3.7	2.4	1.4	1.3	1.2	1.1	0.0	3.1	3.2	3.3
−15	3.5	3.6	3.7	2.4	1.4	1.3	1.2	1.1	0.0	3.1	3.2	3.3	3.4	0.4	1.7	1.6	1.5	2.1	2.5	2.8	2.10
−16	3.1	3.2	3.3	3.4	0.4	1.7	1.6	1.5	2.1	2.5	2.8	2.10	0.7	0.3	3.10	1.9	1.8	2.2	2.6	2.9	0.9
−17	2.5	2.8	2.10	0.7	0.3	3.10	1.9	1.8	2.2	2.6	2.9	0.9	0.6	0.2	3.8	3.9	1.10	2.3	2.7	0.10	0.8
−18	2.6	2.9	0.9	0.6	0.2	3.8	3.9	1.10	2.3	2.7	0.10	0.8	0.5	0.1	3.5	3.6	3.7	2.4	1.4	1.3	1.2
−19	2.7	0.10	0.8	0.5	0.1	3.5	3.6	3.7	2.4	1.4	1.3	1.2	1.1	0.0	3.1	3.2	3.3	3.4	0.4	1.7	1.6
−20	1.4	1.3	1.2	1.1	0.0	3.1	3.2	3.3	3.4	0.4	1.7	1.6	1.5	2.1	2.5	2.8	2.10	0.7	0.3	3.10	1.9

mod $4+5i$

A \ B	−1	−2	−3	−4	−5	−6	−7	−8	−9	−10	−11	−12	−13	−14	−15	−16	−17	−18	−19	−20
20	1.10	2.3	2.7	0.10	0.8	0.5	0.1	3.5	3.6	3.7	2.4	1.4	1.3	1.2	1.1	0.0	3.1	3.2	3.3	3.4
19	3.7	2.4	1.4	1.3	1.2	1.1	0.0	3.1	3.2	3.3	3.4	0.4	1.7	1.6	1.5	2.1	2.5	2.8	2.10	0.7
18	3.3	3.4	0.4	1.7	1.6	1.5	2.1	2.5	2.8	2.10	0.7	0.3	3.10	1.9	1.8	2.2	2.6	2.9	0.9	0.6
17	2.10	0.7	0.3	3.10	1.9	1.8	2.2	2.6	2.9	0.9	0.6	0.2	3.8	3.9	1.10	2.3	2.7	0.10	0.8	0.5
16	0.9	0.6	0.2	3.8	3.9	1.10	2.3	2.7	0.10	0.8	0.5	0.1	3.5	3.6	3.7	2.4	1.4	1.3	1.2	1.1
15	0.8	0.5	0.1	3.5	3.6	3.7	2.4	1.4	1.3	1.2	1.1	0.0	3.1	3.2	3.3	3.4	0.4	1.7	1.6	1.5
14	1.2	1.1	0.0	3.1	3.2	3.3	3.4	0.4	1.7	1.6	1.5	2.1	2.5	2.8	2.10	0.7	0.3	3.10	1.9	1.8
13	1.6	1.5	2.1	2.5	2.8	2.10	0.7	0.3	3.10	1.9	1.8	2.2	2.6	2.9	0.9	0.6	0.2	3.8	3.9	1.10
12	1.9	1.8	2.2	2.6	2.9	0.9	0.6	0.2	3.8	3.9	1.10	2.3	2.7	0.10	0.8	0.5	0.1	3.5	3.6	3.7
11	3.9	1.10	2.3	2.7	0.10	0.8	0.5	0.1	3.5	3.6	3.7	2.4	1.4	1.3	1.2	1.1	0.0	3.1	3.2	3.3
10	3.6	3.7	2.4	1.4	1.3	1.2	1.1	0.0	3.1	3.2	3.3	3.4	0.4	1.7	1.6	1.5	2.1	2.5	2.8	2.10
9	3.2	3.3	3.4	0.4	1.7	1.6	1.5	2.1	2.5	2.8	2.10	0.7	0.3	3.10	1.9	1.8	2.2	2.6	2.9	0.9

Continuation of table.

mod $4+5i$

A \ B	−1	−2	−3	−4	−5	−6	−7	−8	−9	−10	−11	−12	−13	−14	−15	−16	−17	−18	−19	−20
8	2.8	2.10	0.7	0.3	3.10	1.9	1.8	2.2	2.6	2.9	0.9	0.6	0.2	3.8	3.9	1.10	2.3	2.7	0.10	0.8
7	2.9	0.9	0.6	0.2	3.8	3.9	1.10	2.3	2.7	0.10	0.8	0.5	0.1	3.5	3.6	3.7	2.4	1.4	1.3	1.2
6	0.10	0.8	0.5	0.1	3.5	3.6	3.7	2.4	1.4	1.3	1.2	1.1	0.0	3.1	3.2	3.3	3.4	0.4	1.7	1.6
5	1.3	1.2	1.1	0.0	3.1	3.2	3.3	3.4	0.4	1.7	1.6	1.5	2.1	2.5	2.8	2.10	0.7	0.3	3.10	1.9
4	1.7	1.6	1.5	2.1	2.5	2.8	2.10	0.7	0.3	3.10	1.9	1.8	2.2	2.6	2.9	0.9	0.6	0.2	3.8	3.9
3	3.10	1.9	1.8	2.2	2.6	2.9	0.9	0.6	0.2	3.8	3.9	1.10	2.3	2.7	0.10	0.8	0.5	0.1	3.5	3.6
2	3.8	3.9	1.10	2.3	2.7	0.10	0.8	0.5	0.1	3.5	3.6	3.7	2.4	1.4	1.3	1.2	1.1	0.0	3.1	3.2
1	3.5	3.6	3.7	2.4	1.4	1.3	1.2	1.1	0.0	3.1	3.2	3.3	3.4	0.4	1.7	1.6	1.5	2.1	2.5	2.8
0	3.1	3.2	3.3	3.4	0.4	1.7	1.6	1.5	2.1	2.5	2.8	2.10	0.7	0.3	3.10	1.9	1.8	2.2	2.6	2.9
−1	2.5	2.8	2.10	0.7	0.3	3.10	1.9	1.8	2.2	2.6	2.9	0.9	0.6	0.2	3.8	3.9	1.10	2.3	2.7	0.10
−2	2.6	2.9	0.9	0.6	0.2	3.8	3.9	1.10	2.3	2.7	0.10	0.8	0.5	0.1	3.5	3.6	3.7	2.4	1.4	1.3
−3	2.7	0.10	0.8	0.5	0.1	3.5	3.6	3.7	2.4	1.4	1.3	1.2	1.1	0.0	3.1	3.2	3.3	3.4	0.4	1.7
−4	1.4	1.3	1.2	1.1	0.0	3.1	3.2	3.3	3.4	0.4	1.7	1.6	1.5	2.1	2.5	2.8	2.10	0.7	0.3	3.10
−5	0.4	1.7	1.6	1.5	2.1	2.5	2.8	2.10	0.7	0.3	3.10	1.9	1.8	2.2	2.6	2.9	0.9	0.6	0.2	3.8
−6	0.3	3.10	1.9	1.8	2.2	2.6	2.9	0.9	0.6	0.2	3.8	3.9	1.10	2.3	2.7	0.10	0.8	0.5	0.1	3.5
−7	0.2	3.8	3.9	1.10	2.3	2.7	0.10	0.8	0.5	0.1	3.5	3.6	3.7	2.4	1.4	1.3	1.2	1.1	0.0	3.1
−8	0.1	3.5	3.6	3.7	2.4	1.4	1.3	1.2	1.1	0.0	3.1	3.2	3.3	3.4	0.4	1.7	1.6	1.5	2.1	2.5
−9	0.0	3.1	3.2	3.3	3.4	0.4	1.7	1.6	1.5	2.1	2.5	2.8	2.10	0.7	0.3	3.10	1.9	1.8	2.2	2.6
−10	2.1	2.5	2.8	2.10	0.7	0.3	3.10	1.9	1.8	2.2	2.6	2.9	0.9	0.6	0.2	3.8	3.9	1.10	2.3	2.7
−11	2.2	2.6	2.9	0.9	0.6	0.2	3.8	3.9	1.10	2.3	2.7	0.10	0.8	0.5	0.1	3.5	3.6	3.7	2.4	1.4
−12	2.3	2.7	0.10	0.8	0.5	0.1	3.5	3.6	3.7	2.4	1.4	1.3	1.2	1.1	0.0	3.1	3.2	3.3	3.4	0.4
−13	2.4	1.4	1.3	1.2	1.1	0.0	3.1	3.2	3.3	3.4	0.4	1.7	1.6	1.5	2.1	2.5	2.8	2.10	0.7	0.3
−14	3.4	0.4	1.7	1.6	1.5	2.1	2.5	2.8	2.10	0.7	0.3	3.10	1.9	1.8	2.2	2.6	2.9	0.9	0.6	0.2
−15	0.7	0.3	3.10	1.9	1.8	2.2	2.6	2.9	0.9	0.6	0.2	3.8	3.9	1.10	2.3	2.7	0.10	0.8	0.5	0.1
−16	0.6	0.2	3.8	3.9	1.10	2.3	2.7	0.10	0.8	0.5	0.1	3.5	3.6	3.7	2.4	1.4	1.3	1.2	1.1	0.0
−17	0.5	0.1	3.5	3.6	3.7	2.4	1.4	1.3	1.2	1.1	0.0	3.1	3.2	3.3	3.4	0.4	1.7	1.6	1.5	2.1
−18	1.1	0.0	3.1	3.2	3.3	3.4	0.4	1.7	1.6	1.5	2.1	2.5	2.8	2.10	0.7	0.3	3.10	1.9	1.8	2.2
−19	1.5	2.1	2.5	2.8	2.10	0.7	0.3	3.10	1.9	1.8	2.2	2.6	2.9	0.9	0.6	0.2	3.8	3.9	1.10	2.3
−20	1.8	2.2	2.6	2.9	0.9	0.6	0.2	3.8	3.9	1.10	2.3	2.7	0.10	0.8	0.5	0.1	3.5	3.6	3.7	2.4

mod 4—5i

A \ B	20	19	18	17	16	15	14	13	12	11	10	9	8	7	6	5	4	3	2	1
20	1.4	1.3	1.2	1.1	0.0	3.1	3.2	3.3	3.4	2.4	0.10	0.8	0.5	0.1	3.5	3.6	3.7	1.10	2.3	2.7
19	3.10	1.7	1.6	1.5	2.1	2.5	2.8	2.10	0.4	1.4	1.3	1.2	1.1	0.0	3.1	3.2	3.3	3.4	2.4	0.10
18	3.8	3.9	1.9	1.8	2.2	2.6	2.9	0.7	0.3	3.10	1.7	1.6	1.5	2.1	2.5	2.8	2.10	0.4	1.4	1.3
17	3.5	3.6	3.7	1.10	2.3	2.7	0.9	0.6	0.2	3.8	3.9	1.9	1.8	2.2	2.6	2.9	0.7	0.3	3.10	1.7
16	3.1	3.2	3.3	3.4	2.4	0.10	0.8	0.5	0.1	3.5	3.6	3.7	1.10	2.3	2.7	0.9	0.6	0.2	3.8	3.9
15	2.5	2.8	2.10	0.4	1.4	1.3	1.2	1.1	0.0	3.1	3.2	3.3	3.4	2.4	0.10	0.8	0.5	0.1	3.5	3.6
14	2.6	2.9	0.7	0.3	3.10	1.7	1.6	1.5	2.1	2.5	2.8	2.10	0.4	1.4	1.3	1.2	1.1	0.0	3.1	3.2
13	2.7	0.9	0.6	0.2	3.8	3.9	1.9	1.8	2.2	2.6	2.9	0.7	0.3	3.10	1.7	1.6	1.5	2.1	2.5	2.8
12	0.10	0.8	0.5	0.1	3.5	3.6	3.7	1.10	2.3	2.7	0.9	0.6	0.2	3.8	3.9	1.9	1.8	2.2	2.6	2.9
11	1.8	1.2	1.1	0.0	3.1	3.2	3.3	3.4	2.4	0.10	0.8	0.5	0.1	3.5	3.6	3.7	1.10	2.3	2.7	0.9
10	1.7	1.6	1.5	2.1	2.5	2.8	2.10	0.4	1.4	1.3	1.2	1.1	0.0	3.1	3.2	3.3	3.4	2.4	0.10	0.8
9	3.9	1.9	1.8	2.2	2.6	2.9	0.7	0.3	3.10	1.7	1.6	1.5	2.1	2.5	2.8	2.10	0.4	1.4	1.3	1.2
8	3.6	3.7	1.10	2.3	2.7	0.9	0.6	0.2	3.8	3.9	1.9	1.8	2.2	2.6	2.9	0.7	0.3	3.10	1.7	1.6
7	3.2	3.3	3.4	2.4	0.10	0.8	0.5	0.1	3.5	3.6	3.7	1.10	2.3	2.7	0.9	0.6	0.2	3.8	3.9	1.9
6	2.8	2.10	0.4	1.4	1.3	1.2	1.1	0.0	3.1	3.2	3.3	3.4	2.4	0.10	0.8	0.5	0.1	3.5	3.6	3.7
5	2.9	0.7	0.3	3.10	1.7	1.6	1.5	2.1	2.5	2.8	2.10	0.4	1.4	1.3	1.2	1.1	0.0	3.1	3.2	3.3
4	0.9	0.6	0.2	3.8	3.9	1.9	1.8	2.2	2.6	2.9	0.7	0.3	3.10	1.7	1.6	1.5	2.1	2.5	2.8	2.10
3	0.8	0.5	0.1	3.5	3.6	3.7	1.10	2.3	2.7	0.9	0.6	0.2	3.8	3.9	1.9	1.8	2.2	2.6	2.9	0.7
2	1.2	1.1	0.0	3.1	3.2	3.3	3.4	2.4	0.10	0.8	0.5	0.1	3.5	3.6	3.7	1.10	2.3	2.7	0.9	0.6
1	1.6	1.5	2.1	2.5	2.8	2.10	0.4	1.4	1.3	1.2	1.1	0.0	3.1	3.2	3.3	3.4	2.4	0.10	0.8	0.5
0	1.9	1.8	2.2	2.6	2.9	0.7	0.3	3.10	1.7	1.6	1.5	2.1	2.5	2.8	2.10	0.4	1.4	1.3	1.2	1.1
−1	3.7	1.10	2.3	2.7	0.9	0.6	0.2	3.8	3.9	1.9	1.8	2.2	2.6	2.9	0.7	0.3	3.10	1.7	1.6	1.5
−2	3.3	3.4	2.4	0.10	0.8	0.5	0.1	3.5	3.6	3.7	1.10	2.3	2.7	0.9	0.6	0.2	3.8	3.9	1.9	1.8
−3	2.10	0.4	1.4	1.3	1.2	1.1	0.0	3.1	3.2	3.3	3.4	2.4	0.10	0.8	0.5	0.1	3.5	3.6	3.7	1.10
−4	0.7	0.3	3.10	1.7	1.6	1.5	2.1	2.5	2.8	2.10	0.4	1.4	1.3	1.2	1.1	0.0	3.1	3.2	3.3	3.4
−5	0.6	0.2	3.8	3.9	1.9	1.8	2.2	2.6	2.9	0.7	0.3	3.10	1.7	1.6	1.5	2.1	2.5	2.8	2.10	0.4
−6	0.5	0.1	3.5	3.6	3.7	1.10	2.3	2.7	0.9	0.6	0.2	3.8	3.9	1.9	1.8	2.2	2.6	2.9	0.7	0.3
−7	1.1	0.0	3.1	3.2	3.3	3.4	2.4	0.10	0.8	0.5	0.1	3.5	3.6	3.7	1.10	2.3	2.7	0.9	0.6	0.2
−8	1.5	2.1	2.5	2.8	2.10	0.4	1.4	1.3	1.2	1.1	0.0	3.1	3.2	3.3	3.4	2.4	0.10	0.8	0.5	0.1

Continuation of table.

mod 4—5i

A \ B	20	19	18	17	16	15	14	13	12	11	10	9	8	7	6	5	4	3	2	1
−9	1.8	2.2	2.6	2.9	0.7	0.3	3.10	1.7	1.6	1.5	2.1	2.5	2.8	2.10	0.4	1.4	1.3	1.2	1.1	0.0
−10	1.10	2.3	2.7	0.9	0.6	0.2	3.8	3.9	1.9	1.8	2.2	2.6	2.9	0.7	0.3	3.10	1.7	1.6	1.5	2.1
−11	3.4	2.4	0.10	0.8	0.5	0.1	3.5	3.6	3.7	1.10	2.3	2.7	0.9	0.6	0.2	3.8	3.9	1.9	1.8	2.2
−12	0.4	1.4	1.3	1.2	1.1	0.0	3.1	3.2	3.3	3.4	2.4	0.10	0.8	0.5	0.1	3.5	3.6	3.7	1.10	2.3
−13	0.3	3.10	1.7	1.6	1.5	2.1	2.5	2.8	2.10	0.4	1.4	1.3	1.2	1.1	0.0	3.1	3.2	3.3	3.4	2.4
−14	0.2	3.8	3.9	1.9	1.8	2.2	2.6	2.9	0.7	0.3	3.10	1.7	1.6	1.5	2.1	2.5	2.8	2.10	0.4	1.4
−15	0.1	3.5	3.6	3.7	1.10	2.3	2.7	0.9	0.6	0.2	3.8	3.9	1.9	1.8	2.2	2.6	2.9	0.7	0.3	3.10
−16	0.0	3.1	3.2	3.3	3.4	2.4	0.10	0.8	0.5	0.1	3.5	3.6	3.7	1.10	2.3	2.7	0.9	0.6	0.2	3.8
−17	2.1	2.5	2.8	2.10	0.4	1.4	1.3	1.2	1.1	0.0	3.1	3.2	3.3	3.4	2.4	0.10	0.8	0.5	0.1	3.5
−18	2.2	2.6	2.9	0.7	0.3	3.10	1.7	1.6	1.5	2.1	2.5	2.8	2.10	0.4	1.4	1.3	1.2	1.1	0.0	3.1
−19	2.3	2.7	0.9	0.6	0.2	3.8	3.9	1.9	1.8	2.2	2.6	2.9	0.7	0.3	3.10	1.7	1.6	1.5	2.1	2.5
−20	2.4	0.10	0.8	0.5	0.1	3.5	3.6	3.7	1.10	2.3	2.7	0.9	0.6	0.2	3.8	3.9	1.9	1.8	2.2	2.6

mod 4—5i

A \ B	0	−1	−2	−3	−4	−5	−6	−7	−8	−9	−10	−11	−12	−13	−14	−15	−16	−17	−18	−19	−20
20	0.9	0.6	0.2	3.8	2.9	1.9	1.8	2.2	2.6	2.9	0.7	0.3	3.10	1.7	1.6	1.5	2.1	2.5	2.8	2.10	0.4
19	0.8	0.5	0.1	3.5	3.6	3.7	1.10	2.3	2.7	0.9	0.6	0.2	3.8	3.9	1.9	1.8	2.2	2.6	2.9	0.7	0.3
18	1.2	1.1	0.0	3.1	3.2	3.3	3.4	2.4	0.10	0.8	0.5	0.1	3.5	3.6	3.7	1.10	2.3	2.7	0.9	0.6	0.2
17	1.6	1.5	2.1	2.5	2.8	2.10	0.4	1.4	1.3	1.2	1.1	0.0	3.1	3.2	3.3	3.4	2.4	0.10	0.8	0.5	0.1
16	1.9	1.8	2.2	2.6	2.9	0.7	0.3	3.10	1.7	1.6	1.5	2.1	2.5	2.8	2.10	0.4	1.4	1.3	1.2	1.1	0.0
15	3.7	1.10	2.3	2.7	0.9	0.6	0.2	3.8	3.9	1.9	1.8	2.2	2.6	2.9	0.7	0.3	3.10	1.7	1.6	1.5	2.1
14	3.3	3.4	2.4	0.10	0.8	0.5	0.1	3.5	3.6	3.7	1.10	2.3	2.7	0.9	0.6	0.2	3.8	3.9	1.9	1.8	2.2
13	2.10	0.4	1.4	1.3	1.2	1.1	0.0	3.1	3.2	3.3	3.4	2.4	0.10	0.8	0.5	0.1	3.5	3.6	3.7	1.10	2.3
12	0.7	0.3	3.10	1.7	1.6	1.5	2.1	2.5	2.8	2.10	0.4	1.4	1.3	1.2	1.1	0.0	3.1	3.2	3.3	3.4	2.4
11	0.6	0.2	3.8	3.9	1.9	1.8	2.2	2.6	2.9	0.7	0.3	3.10	1.7	1.6	1.5	[illegible]	2.5	2.8	2.10	0.4	1.4
10	0.5	0.1	3.5	3.6	3.7	1.10	2.3	2.7	0.9	0.6	0.2	3.8	3.9	1.9	1.8	2.2	2.6	2.9	0.7	0.3	3.10
9	1.1	0.0	3.1	3.2	3.3	3.4	2.4	0.10	0.8	0.5	0.1	3.5	3.6	3.7	1.10	2.3	2.7	0.9	0.6	0.2	3.8

Continuation of table.

mod 4—5i

A \ B	0	−1	−2	−3	−4	−5	−6	−7	−8	−9	−10	−11	−12	−13	−14	−15	−16	−17	−18	−19	−20
8	1.5	2.1	2.5	2.8	2.10	0.4	1.4	1.3	1.2	1.1	0.0	3.1	3.2	3.3	3.4	2.4	0.10	0.8	0.5	0.1	3.5
7	1.8	2.2	2.6	2.9	0.7	0.3	3.10	1.7	1.6	1.5	2.1	2.5	2.8	2.10	0.4	1.4	1.3	1.2	1.1	0.0	3.1
6	1.10	2.3	2.7	0.9	0.6	0.2	3.8	3.9	1.9	1.8	2.2	2.6	2.9	0.7	0.3	3.10	1.7	1.6	1.5	2.1	2.5
5	3.4	2.4	0.10	0.8	0.5	0.1	3.5	3.6	3.7	1.10	2.3	2.7	0.9	0.6	0.2	0.8	3.9	1.9	1.8	2.2	2.6
4	0.4	1.4	1.3	1.2	1.1	0.0	3.1	3.2	3.3	3.4	2.4	0.10	0.8	0.5	0.1	3.5	3.6	3.7	1.10	2.3	2.7
3	0.3	3.10	1.7	1.6	1.5	2.1	2.5	2.8	2.10	0.4	1.4	1.3	1.2	1.1	0.0	3.1	3.2	3.3	3.4	2.4	0.10
2	0.2	3.8	3.9	1.9	1.8	2.2	2.6	2.9	0.7	0.3	3.10	1.7	1.6	1.5	2.1	2.5	2.8	2.10	0.4	1.4	1.3
1	0.1	3.5	3.6	3.7	1.10	2.3	2.7	0.9	0.6	0.2	3.8	3.9	1.9	1.8	2.2	2.6	2.9	0.7	0.3	3.10	1.7
0	0.0	3.1	3.2	3.3	3.4	2.4	0.10	0.8	0.5	0.1	3.5	3.6	3.7	1.10	2.3	2.7	0.9	0.6	0.2	3.8	3.9
−1	2.1	2.5	2.8	2.10	0.4	1.4	1.3	1.2	1.1	0.0	3.1	3.2	3.3	3.4	2.4	0.10	0.8	0.5	0.1	3.5	3.6
−2	2.2	2.6	2.9	0.7	0.3	3.10	1.7	1.6	1.5	2.1	2.5	2.8	2.10	0.4	1.4	1.3	1.2	1.1	0.0	3.1	3.2
−3	2.3	2.7	0.9	0.6	0.2	3.8	3.9	1.9	1.8	2.2	2.6	2.9	0.7	0.3	3.10	1.7	1.6	1.5	2.1	2.5	2.8
−4	2.4	0.10	0.8	0.5	0.1	3.5	3.6	3.7	1.10	2.3	2.7	0.9	0.6	0.2	3.8	3.9	1.9	1.8	2.2	2.6	2.9
−5	1.4	1.3	1.2	1.1	0.0	3.1	3.2	3.3	3.4	2.4	0.10	0.8	0.5	0.1	3.5	3.6	3.7	1.10	2.3	2.7	0.9
−6	3.10	1.7	1.6	1.5	2.1	2.5	2.8	2.10	0.4	1.4	1.3	1.2	1.1	0.0	3.1	3.2	3.3	3.4	2.4	0.10	0.8
−7	3.8	3.9	1.9	1.8	2.2	2.6	2.9	0.7	0.3	3.10	1.7	1.6	1.5	2.1	2.5	2.8	2.10	0.4	1.4	1.3	1.2
−8	3.5	3.6	3.7	1.10	2.3	2.7	0.9	0.6	0.2	3.8	3.9	1.9	1.8	2.2	2.6	2.9	0.7	0.3	3.10	1.7	1.6
−9	3.1	3.2	3.3	3.4	2.4	0.10	0.8	0.5	0.1	3.5	3.6	3.7	1.10	2.3	2.7	0.9	0.6	0.2	3.8	3.9	1.9
−10	2.5	2.8	2.10	0.4	1.4	1.3	1.2	1.1	0.0	3.1	3.2	3.3	3.4	2.4	0.10	0.8	0.5	0.1	3.5	3.6	3.7
−11	2.6	2.9	0.7	0.3	3.10	1.7	1.6	1.5	2.1	2.5	2.8	2.10	0.4	1.4	1.3	1.2	1.1	0.0	3.1	3.2	3.3
−12	2.7	0.9	0.6	0.2	3.8	3.9	1.9	1.8	2.2	2.6	2.9	0.7	0.3	3.10	1.7	1.6	1.5	2.1	2.5	2.8	2.10
−13	0.10	0.8	0.5	0.1	3.5	3.6	3.7	1.10	2.3	2.7	0.9	0.6	0.2	3.8	3.9	1.9	1.8	2.2	2.6	2.9	0.7
−14	1.3	1.2	1.1	0.0	3.1	3.2	3.3	3.4	2.4	0.10	0.8	0.5	0.1	3.5	3.6	3.7	1.10	2.3	2.7	0.9	0.6
−15	1.7	1.6	1.5	2.1	2.5	2.8	2.10	0.4	1.4	1.3	1.2	1.1	0.0	3.1	3.2	3.3	3.4	2.4	0.10	0.8	0.5
−16	3.9	1.9	1.8	2.2	2.6	2.9	0.7	0.3	3.10	1.7	1.6	1.5	2.1	2.5	2.8	2.10	0.4	1.4	1.3	1.2	1.1
−17	3.6	3.7	1.10	2.3	2.7	0.9	0.6	0.2	3.8	3.9	1.9	1.8	2.2	2.6	2.9	0.7	0.3	3.10	1.7	1.6	1.5
−18	3.2	3.3	3.4	2.4	0.10	0.8	0.5	0.1	3.5	3.6	3.7	1.10	2.3	2.7	0.9	0.6	0.2	3.8	3.9	1.9	1.8
−19	2.8	2.10	0.4	1.4	1.3	1.2	1.1	0.0	3.1	3.2	3.3	3.4	2.4	0.10	0.8	0.5	0.1	3.5	3.6	3.7	1.10
−20	2.9	0.7	0.3	3.10	1.7	1.6	1.5	2.1	2.5	2.8	2.10	0.4	1.4	1.3	1.2	1.1	0.0	3.1	3.2	3.3	3.4

ATTACHMENT 2

RECODING TABLES

$< 1 - w \mid \bar{13} \mid ^{+}_{1+4l}$

r_1 \ r_2	0	1	2	3	4	5	6	7	8	9	10	11	12
0	0.0	1.4	0.2	3.2	1.3	2.4	0.3	2.3	0.4	3.3	1.2	2.2	3.4
1	0.3	2.1	0.4	0.1	1.2	0.2	3.4	1.1	2.4	0.3	3.1	1.4	3.3
2	3.1	1.1	2.2	0.3	0.0	1.4	0.1	3.2	1.3	3.4	1.1	2.3	0.2
3	0.1	2.3	1.3	3.3	1.1	2.1	0.4	0.0	1.2	0.2	3.2	2.1	2.4
4	3.4	0.0	2.4	0.2	3.1	1.3	2.2	0.3	2.1	1.4	3.3	1.2	2.2
5	3.3	3.2	2.1	3.4	0.1	2.3	0.2	3.3	1.1	2.4	0.4	3.1	1.4
6	0.4	3.1	1.2	2.2	3.2	0.0	2.4	0.1	3.2	1.3	3.4	0.3	2.3
7	2.4	0.3	2.3	1.4	3.3	1.2	2.1	0.4	0.0	1.2	0.2	3.2	1.1
8	1.3	3.4	1.1	2.4	0.4	3.1	1.3	2.2	0.3	2.1	1.4	0.1	1.2
9	1.4	0.2	3.2	1.3	3.4	0.1	2.3	0.2	3.3	1.1	2.2	0.4	0.0
10	2.1	0.4	0.1	1.2	2.2	3.2	0.0	2.4	0.1	3.1	1.3	3.3	0.3
11	1.1	2.2	0.3	3.1	1.4	3.3	1.2	2.1	3.4	0.0	2.3		3.1
12	2.3	1.3	3.4	1.1	2.3	0.4	3.1	1.4	2.2	3.2	2.1	[illegible]	0.1

$< 1 - w \mid \bar{13} \mid ^{+}_{1-4l}$

r_1 \ r_2	0	1	2	3	4	5	6	7	8	9	10	11	12
0	0.0	3.3	1.1	0.1	0.4	0.3	3.4	1.4	2.3	2.4	2.1	3.1	1.3
1	2.4	2.1	3.1	1.3	0.0	1.2	1.1	3.3	0.4	0.2	3.4	2.2	2.3
2	0.2	3.4	2.2	2.3	1.4	2.1	3.2	1.3	3.1	1.2	0.1	3.3	0.4
3	1.2	0.1	3.3	0.3	0.2	0.4	2.2	2.4	1.4	2.3	3.2	1.1	3.1
4	2.3	3.2	0.0	3.1	1.1	0.1	1.2	0.3	3.4	0.4	2.2	2.4	1.3
5	1.4	0.2	2.4	2.1	2.3	1.3	0.0	3.2	1.1	0.1	1.2	0.3	3.4
6	3.3	0.4	0.1	3.4	2.2	0.2	1.4	2.1	2.3	1.3	0.0	3.2	1.1
7	1.3	3.1	1.2	0.0	3.3	0.3	0.1	3.4	2.2	0.2	1.4	2.1	2.4
8	3.4	1.4	2.3	3.2	2.1	3.1	1.2	0.0	3.3	0.3	0.1	0.4	2.2
9	0.3	3.3	0.4	0.2	2.4	1.4	2.3	3.2	2.1	3.1	1.1	0.0	1.2
10	3.2	1.1	3.1	1.2	0.3	3.4	0.4	0.2	2.4	2.2	2.3	1.3	2.1
11	2.2	2.4	1.3	2.1	3.2	1.1	3.3	1.2	0.1	3.4	0.3	0.2	1.4
12	0.4	0.3	0.2	1.4	2.2	2.4	1.3	3.1	3.2	0.0	3.3	1.1	0.1

$< 1 - w \mid \bar{12} \mid ^{+}_{3+4l}$

r_1 \ r_2	0	1	2	3	4	5	6	7	8	9	10	11	12
0	0.0	3.6	3.2	2.2	0.4	0.6	3.4	1.4	2.6	2.4	0.2	1.2	1.6
1	0.3	2.1	3.4	2.6	2.3	1.1	1.2	3.1	1.6	0.6	2.5	0.1	1.5
2	3.6	0.2	2.2	3.1	0.6	3.3	1.4	1.5	2.4	3.5	1.2	1.3	2.6
3	0.6	3.4	0.1	2.3	2.4	1.2	0.5	1.6	3.6	2.5	3.2	2.2	0.3
4	0.2	1.2	3.1	0.0	3.3	2.5	1.5	0.4	3.5	3.4	1.4	2.6	2.3
5	3.3	0.1	1.5	2.4	2.1	0.5	1.3	3.6	1.1	0.2	3.1	1.6	0.6
6	1.2	0.5	0.0	3.6	2.5	2.2	0.4	0.3	3.3	1.4	0.1	2.4	3.5
7	3.2	1.5	0.4	2.1	3.4	1.3	2.3	2.4	0.2	0.5	1.6	0.0	2.5
8	1.3	2.6	3.6	1.1	2.2	3.1	1.6	3.3	2.5	0.1	0.4	3.5	2.1
9	2.2	0.3	0.6	3.4	1.4	1.5	2.4	3.5	0.5	1.3	0.0	1.1	3.2
10	2.6	2.3	0.2	1.2	0.5	1.6	3.6	2.5	3.2	0.4	0.3	2.1	1.4
11	1.6	0.6	3.3	3.2	1.5	0.4	3.5	3.4	1.3	2.6	1.1	0.2	2.2
12	2.3	3.5	2.1	0.5	2.6	3.6	1.1	3.2	3.1	0.3	0.6	1.4	0.1

$< 1 - w \mid \bar{13} \mid ^{+}_{3-4l}$

r_1 \ r_2	0	1	2	3	4	5	6	7	8	9	10	11	12
0	0.0	0.3	0.5	3.5	0.2	3.3	3.2	1.2	3.3	2.2	1.5	2.5	2.3
1	0.5	2.1	0.2	0.4	3.2	0.1	3.6	2.6	1.5	0.3	2.3	3.5	0.6
2	1.2	0.4	2.2	0.1	1.1	2.6	0.0	3.4	0.5	3.5	0.2	1.3	2.1
3	2.2	1.5	1.1	2.3	0.0	1.4	0.5	2.1	3.1	0.4	3.2	1.2	3.6
4	3.4	2.3	3.5	1.4	1.3	2.1	1.6	0.4	2.2	2.4	3.6	2.6	1.5
5	3.5	3.1	1.3	3.2	1.6	3.6	2.2	3.3	1.1	2.6	2.5	3.4	0.5
6	0.4	3.2	2.4	3.6	2.6	3.3	3.4	2.3	2.5	1.4	0.5	0.6	3.1
7	2.4	1.1	2.6	2.5	3.4	0.5	0.3	1.4	1.3	0.6	1.6	0.4	1.2
8	1.5	2.5	1.4	0.5	[illegible]	3.1	1.3	0.2	1.6	3.6	1.2	3.3	1.1
9	1.4	3.5	0.6	1.6	0.4	0.2	2.4	3.6	0.1	3.3	3.4	1.5	0.3
10	0.2	1.6	3.2	1.2	2.4	1.1	0.1	2.5	3.4	0.0	0.3	3.1	3.5
11	3.2	0.1	3.3	2.2	1.5	2.5	1.4	0.0	0.6	3.1	2.1	0.2	2.4
12	2.5	2.6	1.5	0.3	2.3	3.5	0.6	1.6	0.1	1.2	2.4	2.2	0.1

$< \mid -w \mid \overline{13} \mid^{+}_{2+5i}$

r_1 \ r_2	0	1	2	3	4	5	6	7	8	9	10	11	12
0	0.0	3.4	3.7	2.7	1.2	0.4	0.2	2.2	2.4	3.2	0.7	1.7	1.4
1	1.3	2.1	3.2	3.4	1.5	1.4	0.3	0.1	3.7	2.5	2.6	0.6	0.2
2	0.1	1.6	2.2	2.6	3.2	0.5	1.7	1.1	0.0	3.4	3.5	2.7	0.3
3	1.1	0.0	3.6	3.7	2.7	2.6	0.4	0.2	1.3	2.1	3.2	0.5	1.5
4	0.5	1.3	2.1	3.3	3.4	1.5	2.7	0.3	0.1	1.6	2.3	2.6	0.4
5	0.3	0.4	1.6	2.2	3.1	3.2	0.5	1.5	1.1	3.3	3.6	2.4	2.7
6	1.5	1.1	0.3	3.6	3.7	2.3	2.6	0.4	1.4	1.3	3.1	3.3	2.5
7	3.5	0.5	1.3	1.1	3.3	3.4	2.4	0.6	0.3	1.7	1.6	2.3	3.1
8	2.3	0.7	0.4	1.6	1.3	3.1	3.5	2.5	1.2	1.1	0.2	3.6	2.4
9	2.5	2.4	0.6	0.3	3.6	2.1	2.3	0.7	3.5	1.4	1.3	0.1	3.3
10	3.1	3.5	2.5	1.2	0.1	3.3	2.2	2.4	0.6	0.7	1.7	1.6	0.0
11	2.1	2.3	0.7	1.5	1.4	0.0	3.1	3.7	2.5	1.2	0.6	0.2	3.6
12	3.3	2.2	2.6	0.6	0.5	1.7	2.1	2.3	3.4	3.5	1.4	1.2	0.1

$< \mid -w \mid \overline{13} \mid^{+}_{2-5i}$

r_1 \ r_2	0	1	2	3	4	5	6	7	8	9	10	11	12
0	0.0	2.3	0.1	3.1	3.7	3.3	2.7	0.7	1.3	1.7	1.1	2.1	0.3
1	0.5	2.1	2.4	0.0	2.3	3.5	3.1	3.7	3.6	1.5	0.7	1.3	2.2
2	0.6	1.2	2.2	0.4	2.1	2.4	3.3	2.3	3.5	3.4	1.7	3.6	1.4
3	1.6	0.5	1.4	0.6	0.3	2.2	0.4	3.1	2.4	3.3	3.2	2.7	3.4
4	3.2	2.7	1.2	1.6	0.5	1.1	0.6	0.3	2.3	0.4	0.1	2.5	3.7
5	3.5	2.5	3.7	1.4	2.7	1.2	1.3	0.5	1.1	2.1	0.3	0.0	2.6
6	0.2	3.3	2.6	3.5	1.6	3.7	1.4	1.5	0.7	1.3	2.2	1.1	2.1
7	2.2	0.1	3.1	0.2	3.3	2.7	3.5	3.4	1.7	3.6	1.5	0.6	1.3
8	1.5	0.6	0.0	2.3	0.1	3.1	2.5	3.3	3.2	0.7	3.4	1.7	0.5
9	1.2	1.7	0.5	2.1	2.4	0.3	2.3	2.6	3.1	2.5	3.6	3.2	0.7
10	3.6	1.4	0.7	1.2	1.3	0.4	1.1	2.4	0.2	2.3	2.6	3.4	2.5
11	2.6	3.4	1.6	3.7	1.4	1.5	0.3	1.3	0.4	0.1	2.4	0.2	3.2
12	2.5	0.2	3.3	2.7	3.5	1.6	1.7	1.1	1.5	0.3	0.0	0.4	0.1

$< \mid -w \mid \overline{13} \mid^{+}_{1+6i}$

r_1 \ r_2	0	1	2	3	4	5	6	7	8	9	10	11	12
0	0.0	3.3	2.5	1.5	1.6	0.3	0.6	2.6	2.3	3.6	3.5	0.5	1.3
1	2.4	2.1	1.3	2.6	1.8	1.9	0.2	0.5	3.8	3.7	3.3	3.2	0.4
2	1.1	2.5	2.2	1.6	3.8	0.6	0.9	0.1	0.4	3.5	3.4	1.3	0.0
3	2.1	1.4	2.6	2.8	1.9	3.5	0.5	0.8	0.0	1.1	3.2	0.4	1.6
4	1.9	2.2	1.7	3.8	3.7	0.9	3.2	0.4	0.7	2.1	1.3	2.7	1.1
5	1.4	0.9	2.3	0.3	3.5	3.4	0.8	2.7	1.1	3.2	2.2	1.6	2.8
6	2.9	1.7	0.8	3.7	0.2	3.2	3.1	0.7	2.1	1.4	2.7	2.3	1.9
7	0.9	3.9	0.3	0.7	3.4	0.1	2.7	1.1	1.2	2.2	1.7	2.8	3.7
8	3.4	0.8	3.6	0.2	1.2	3.1	0.7	2.8	1.4	1.5	2.3	0.3	2.9
9	3.9	3.1	0.7	3.3	0.1	2.7	2.4	1.2	2.9	1.7	1.8	3.7	0.2
10	0.1	3.6	2.4	1.2	3.1	0.0	2.8	2.5	1.5	3.9	0.3	0.6	3.4
11	3.1	0.0	3.3	1.4	1.5	2.4	2.1	2.9	2.6	1.8	3.6	0.2	0.5
12	0.4	2.4	1.2	1.8	1.7	1.8	2.5	2.2	3.9	3.8	0.6	3.3	0.1

$< \mid -w \mid \overline{13} \mid^{+}_{1-6i}$

r_1 \ r_2	0	1	2	3	4	5	6	7	8	9	10	11	12
0	0.0	1.4	3.1	2.1	2.9	2.4	1.9	3.9	0.4	0.9	0.1	1.1	3.4
1	1.3	2.1	1.6	2.4	2.2	3.9	2.5	0.9	3.7	1.1	0.8	0.0	1.4
2	1.6	3.3	2.2	1.8	2.5	2.3	3.7	0.3	0.8	3.5	1.4	1.3	3.2
3	2.6	1.8	2.8	2.3	0.7	0.3	0.5	3.5	0.2	1.3	3.2	2.4	3.3
4	2.8	2.7	0.7	2.9	0.5	0.6	0.2	0.4	3.2	0.1	1.1	2.6	2.5
5	0.3	2.9	3.8	0.6	3.9	0.4	1.2	0.1	1.1	1.5	0.0	1.4	2.7
6	3.8	0.2	3.9	3.6	1.2	3.7	1.1	1.5	3.5	1.4	1.7	2.1	1.6
7	1.8	3.6	0.1	3.7	3.4	1.5	3.5	3.1	1.7	3.2	1.6	1.9	2.2
8	2.3	0.7	3.4	0.0	3.5	3.1	2.1	3.2	2.4	1.9	2.6	1.8	0.9
9	0.8	0.5	0.6	3.1	2.1	1.2	2.4	2.2	2.6	2.5	0.9	2.7	0.7
10	0.6	1.3	0.4	1.2	3.3	2.2	1.5	2.5	2.3	2.7	0.3	0.8	3.8
11	3.6	1.2	3.3	3.4	1.5	2.8	2.3	1.7	0.3	0.5	3.8	0.2	1.3
12	1.3	3.4	0.0	2.8	3.1	1.7	2.9	0.5	1.9	0.2	0.4	3.6	0.1

$< \mid -w \mid _{\overline{13}} \mid ^{+}_{4+5i}$

$r_1 \backslash r_2$	0	1	2	3	4	5	6	7	8	9	10	11	12
0	0.0	0.1	3.4	2.4	0.9	1.1	3.9	1.9	3.1	2.9	0.4	1.4	2.1
1	3.8	2.1	0.0	0.7	3.4	0.8	1.5	3.6	3.9	2.5	0.10	0.3	0.4
2	0.3	3.5	2.2	2.1	0.6	0.7	1.2	1.8	3.2	3.6	2.6	1.3	0.10
3	1.3	0.2	3.1	2.3	2.2	0.5	0.6	1.6	1.10	2.8	3.2	3.6	1.7
4	3.10	1.7	0.1	2.5	2.4	2.8	1.1	0.5	1.9	3.7	1.10	2.8	3.2
5	2.8	3.8	3.10	0.0	2.6	3.4	2.4	1.5	1.1	0.5	3.3	3.7	2.9
6	0.10	2.9	3.5	3.8	2.1	2.7	0.7	3.4	0.2	1.5	1.1	2.10	3.3
7	2.10	1.3	0.10	3.1	3.5	2.2	1.4	2.7	0.7	0.1	1.8	1.5	0.9
8	0.8	0.9	1.7	1.3	2.5	3.1	3.5	0.4	1.4	0.6	0.0	1.10	1.8
9	1.10	1.2	0.8	3.10	1.7	3.9	2.5	3.1	0.3	0.4	0.5	2.1	3.7
10	3.3	3.7	1.6	1.2	0.8	3.10	3.6	2.6	2.5	0.2	0.3	1.1	2.2
11	2.3	2.10	3.3	0.6	1.6	1.2	3.8	3.2	2.7	2.6	0.[illegible]	0.2	1.5
12	1.8	2.4	2.3	2.10	0.5	1.9	1.6	3.5	2.8	1.4	2.7	0.0	0.1

$< \mid -w \mid _{\overline{13}} \mid ^{+}_{4-5i}$

$r_1 \backslash r_2$	0	1	2	3	4	5	6	7	8	9	10	11	12
0	0.0	0.6	0.3	3.3	0.7	1.6	3.7	1.7	3.6	2.7	1.3	2.3	2.6
1	0.1	2.1	0.5	0.2	2.10	0.6	1.9	3.3	3.9	3.2	0.10	1.7	2.4
2	1.4	0.0	2.2	1.1	0.1	0.7	0.5	3.7	2.10	3.6	2.8	1.3	2.3
3	2.4	3.10	2.1	2.3	1.5	0.0	0.6	1.1	3.3	0.7	3.2	3.8	1.7
4	3.9	1.4	3.8	2.2	2.4	1.8	2.1	0.5	1.5	2.10	3.6	2.8	3.5
5	3.1	3.6	3.10	3.5	2.3	1.4	1.10	2.2	1.1	0.9	0.7	3.2	2.9
6	0.9	2.5	3.2	3.8	3.1	2.4	3.10	3.4	0.5	1.5	0.8	0.6	2.8
7	2.9	0.8	2.6	2.8	3.5	2.5	1.4	1.10	0.4	1.1	1.8	1.2	0.5
8	1.1	0.9	1.2	2.7	2.9	3.1	0.2	3.10	3.4	0.3	1.5	1.10	1.6
9	1.9	1.5	0.8	1.6	0.10	3.5	2.5	0.1	3.8	0.4	0.2	1.8	3.4
10	0.4	3.7	1.8	1.2	2.7	1.3	3.1	2.6	0.0	3.5	0.3	0.1	1.10
11	3.4	0.8	3.8	0.8	1.6	0.10	1.7	2.5	2.7	2.1	3.1	0.2	0.0
12	2.1	0.4	3.7	2.10	1.2	1.9	1.8	3.9	2.6	0.10	2.2	2.5	0.1

$< \mid -w \mid _{\overline{17}} \mid ^{+}_{3+4i}$

$r_1 \backslash r_2$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
0	0.0	2.3	0.2	2.1	1.3	3.1	0.6	3.6	3.2	1.2	1.6	2.6	1.1	3.3	0.1	2.2	0.3
1	0.4	2.1	3.3	0.1	2.2	0.3	2.4	1.2	3.4	2.6	1.5	3.5	0.6	1.4	0.5	0.0	2.3
2	3.3	1.1	2.2	0.5	0.0	2.3	0.2	2.5	1.5	3.1	0.6	3.6	3.2	1.2	1.6	0.4	1.6
3	3.5	0.5	1.4	2.3	0.4	2.1	3.3	0.1	1.3	3.6	2.4	1.2	3.4	2.6	1.5	3.1	1.1
4	1.4	3.2	0.4	1.6	3.3	1.1	2.2	0.5	0.0	0.3	3.4	2.5	1.5	3.1	1.3	3.6	2.4
5	2.5	1.6	2.6	1.1	3.5	0.5	1.4	2.3	0.4	2.1	0.2	3.1	1.3	0.1	2.4	0.3	3.4
6	3.1	1.3	3.5	0.6	1.4	3.2	0.4	1.6	3.3	1.1	2.2	0.1	2.2	0.3	0.0	2.5	0.2
7	0.1	2.4	0.3	3.2	1.2	1.6	2.6	1.1	3.5	0.5	1.4	3.2	0.0	2.3	0.2	2.1	1.3
8	0.3	0.0	2.5	0.2	2.6	1.5	3.5	0.6	1.4	3.2	1.2	1.6	2.6	2.1	3.3	0.1	2.2
9	2.3	0.2	2.1	1.3	0.1	0.6	3.6	3.2	1.2	3.4	2.6	1.5	3.5	0.6	2.2	0.5	0.0
10	2.1	3.3	0.1	2.2	0.3	0.0	1.2	3.4	2.5	1.5	3.1	0.6	3.6	3.2	1.2	2.3	0.4
11	1.1	2.2	0.5	0.0	2.3	0.2	2.1	0.2	3.1	1.3	3.6	2.4	1.2	3.4	2.6	1.5	3.3
12	0.5	1.4	2.3	0.4	0.1	3.3	1.1	2.2	0.1	2.4	0.3	3.4	2.5	1.5	3.1	0.6	3.6
13	3.4	0.4	1.6	3.3	1.1	3.5	0.5	1.4	2.3	0.0	2.5	0.2	3.1	1.3	3.6	2.4	1.2
14	1.5	3.1	1.1	3.5	0.6	1.4	3.2	0.4	1.6	3.3	2.1	1.3	0.1	2.4	0.3	3.4	2.5
15	1.3	3.6	2.4	3.6	3.2	1.2	1.6	2.6	1.1	3.6	0.5	2.2	0.3	0.0	2.5	0.2	3.1
16	2.4	0.3	0.0	2.5	3.4	2.6	1.5	3.5	0.6	1.4	3.2	0.4	2.3	0.2	2.1	1.3	0.1

$< \mid -w \mid _{\overline{17}} \mid ^{+}_{3-4i}$

$r_1 \backslash r_2$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
0	0.0	3.4	1.3	0.6	2.4	1.6	1.1	0.1	0.3	2.3	2.1	3.1	3.6	0.4	2.6	3.3	1.4
1	2.3	2.1	3.1	3.6	1.2	2.5	3.3	1.4	0.0	0.2	1.3	2.2	2.4	3.4	1.1	0.5	0.3
2	0.2	1.3	2.2	2.4	3.4	1.5	0.6	0.3	1.6	2.1	0.1	3.6	2.3	2.5	3.1	1.4	0.0
3	2.1	0.1	3.6	2.3	2.5	3.1	3.5	1.2	0.2	3.3	2.2	0.0	3.4	1.3	0.6	2.5	1.6
4	3.3	2.2	0.0	3.4	1.3	0.6	2.4	3.2	1.5	0.1	0.3	2.3	2.1	3.1	3.5	1.2	0.4
5	1.1	0.3	2.3	2.1	3.1	3.6	1.2	2.5	2.6	3.5	0.0	0.2	1.3	0.6	2.4	3.2	1.5
6	3.5	1.4	0.2	1.3	2.2	2.4	3.4	1.5	0.6	0.5	3.2	2.1	3.1	3.6	1.2	2.5	2.6
7	0.5	3.2	1.6	0.1	3.6	2.3	2.5	3.1	3.5	1.2	0.4	0.2	2.2	2.4	3.4	1.5	0.6

Continuation of table.

$\langle \mid -w \mid \overline{17} \mid \overset{+}{3-4i}$

8	1.2	0.4	2.6	3.3	0.0	3.4	1.3	0.6	2.4	3.2	3.3	1.1	0.1	2.3	2.5	3.1	3.5
9	3.2	1.5	1.1	0.5	0.3	2.1	3.1	3.6	1.2	0.4	2.6	3.3	1.4	0.0	1.3	0.6	2.4
10	2.5	2.6	3.5	1.4	0.4	0.2	2.2	2.4	3.2	1.5	1.1	0.5	0.3	1.6	2.1	3.6	1.2
11	1.5	0.6	0.5	3.2	1.6	1.1	0.1	1.2	2.5	2.6	3.5	1.4	0.4	0.2	3.3	2.2	3.4
12	3.1	3.5	1.2	0.4	2.6	3.3	2.2	0.0	1.5	0.6	0.5	3.2	1.6	1.1	0.1	0.3	2.3
13	1.3	2.4	3.2	1.5	1.1	0.1	0.3	2.3	2.1	3.5	1.2	0.4	2.6	3.3	1.4	0.0	0.2
14	0.1	3.6	2.5	2.6	3.3	1.4	0.0	0.2	1.3	2.2	3.2	0.6	1.1	0.5	0.3	1.6	2.1
15	2.2	0.0	3.4	1.1	0.5	0.3	1.6	2.1	0.1	3.6	2.3	2.6	3.5	1.4	0.4	0.2	3.3
16	0.3	2.3	2.5	3.1	1.4	0.4	0.2	3.3	2.2	0.0	3.4	1.3	0.5	3.2	1.6	1.1	0.1

$\langle \mid -w \mid \overline{17} \mid \overset{+}{2+5i}$

r \ r_2	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
0	0.0	1.7	0.3	1.5	0.7	2.5	3.2	2.2	3.3	1.3	0.2	1.2	0.5	2.7	3.5	2.3	3.7
1	3.3	2.1	0.2	1.1	0.5	0.6	3.5	2.6	3.7	3.1	1.6	0.1	1.4	0.4	1.5	0.7	2.4
2	2.5	3.1	2.2	0.1	1.3	0.4	1.2	0.7	2.7	3.4	2.3	3.6	0.0	1.7	0.3	0.5	2.6
3	2.7	3.5	2.3	3.7	0.0	1.6	0.3	1.4	0.6	1.5	3.2	2.4	3.3	2.1	0.2	1.2	0.4
4	0.3	1.5	0.7	2.4	3.4	2.1	3.6	1.1	1.7	1.2	0.5	2.6	2.5	3.1	1.6	0.1	1.4
5	1.7	1.1	0.5	0.6	2.5	3.2	2.2	3.3	1.3	0.2	1.4	0.4	2.7	3.4	2.3	3.6	0.0
6	2.1	0.2	1.3	0.4	1.2	3.5	2.6	3.7	3.1	1.6	0.1	1.7	0.6	1.5	3.2	2.4	3.3
7	3.1	2.2	0.1	1.6	0.3	1.4	0.7	2.7	3.4	2.3	3.6	1.1	0.2	1.2	0.5	2.6	2.5
8	3.5	2.3	3.7	0.0	3.6	1.1	1.7	0.6	1.5	3.2	2.2	3.3	1.3	0.1	1.4	0.4	2.7
9	1.5	0.7	2.4	3.4	2.1	3.3	1.3	0.2	1.2	3.5	2.6	3.7	3.1	1.6	0.0	1.7	0.3
10	1.1	0.5	0.6	2.5	3.2	2.2	3.1	1.6	0.3	1.4	0.7	2.7	3.4	2.3	3.6	2.1	0.2
11	0.1	1.3	0.4	1.2	3.5	2.6	3.7	2.1	3.6	1.1	1.7	0.6	1.5	3.2	2.4	3.3	2.2
12	3.7	0.0	1.6	0.3	1.4	0.7	2.4	3.4	2.2	3.3	1.3	0.2	1.2	0.5	2.6	2.5	3.1
13	2.3	3.4	2.1	3.6	1.1	0.5	0.6	2.5	3.2	3.7	3.1	1.6	0.1	1.4	0.4	2.7	3.5
14	0.7	2.4	3.2	2.2	0.1	1.3	0.4	1.2	3.5	2.6	3.4	2.3	3.6	0.0	1.7	0.3	1.5
15	0.5	0.6	2.5	2.3	3.7	0.0	1.6	0.3	1.4	0.7	2.7	3.2	2.4	3.3	2.1	0.2	1.1
16	1.3	0.4	2.7	3.5	2.4	3.4	2.1	3.6	1.1	1.7	0.6	1.5	2.6	2.5	3.1	2.2	0.1

$\langle \mid -w \mid \overline{17} \mid \overset{+}{2-5i}$

r_1 \ r_2	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
0	0.0	0.3	1.7	1.6	3.3	2.6	2.1	1.1	0.7	2.7	3.1	0.1	0.6	1.3	3.6	3.7	2.3
1	2.6	2.1	1.1	0.7	2.7	3.1	0.2	2.2	1.3	3.6	3.7	2.3	0.0	0.5	1.5	3.4	3.5
2	3.3	0.2	2.2	1.3	3.6	3.7	2.3	0.1	0.6	1.5	3.4	3.5	2.4	2.1	1.2	1.7	1.6
3	2.7	3.1	0.1	0.6	1.5	3.4	3.5	2.4	0.0	0.5	1.7	3.2	3.3	0.4	2.2	1.3	0.7
4	3.6	3.7	2.3	0.0	0.5	1.7	3.2	3.3	0.4	2.1	1.2	0.7	2.5	3.1	0.1	0.6	1.5
5	1.7	3.4	3.5	2,4	2.1	1.2	0.7	0.4	3.1	0.3	2.2	1.4	3.6	3.7	2.3	0.0	0.5
6	1.2	0.7	3.2	3.3	0.4	2.2	1.4	3.6	2.6	2.3	1.1	0.6	1.5	3.4	3.5	2.4	2.1
7	2.2	1.4	3.6	2.5	3.1	0.3	0.6	1.6	3.4	0.2	2.4	2.1	0.5	1.7	3.2	3.3	0.4
8	0.3	0.6	1.6	3.4	2.6	2.3	1.1	0.5	2.7	3.2	3.3	0.4	2.2	1.2	0.7	2.5	3.1
9	2.3	1.1	0.5	2.7	3.2	0.2	2.4	1.3	1.2	0.7	2.5	3.1	0.3	0.6	1.4	3.6	2.6
10	0.2	2.4	1.3	1.2	3.7	2.5	0,1	0.4	2.2	1.4	3.6	2.6	2.3	1.1	0.5	1.6	3.4
11	3.2	0.1	0.4	1.5	1.4	3.5	2.6	3.1	0.3	0.6	1.6	3.4	0.2	2.4	1.3	1.2	2.7
12	3.7	2.5	0.0	0.3	1.7	1.6	3.4	0.2	2.3	1.1	0.5	2.7	3.2	0.1	0.4	1.5	1.4
13	1.6	3.5	2.6	2.1	1.1	0.5	2.7	3.2	0.1	2.4	1.3	1.2	3.7	2.5	0.0	0.3	1.7
14	0.7	2.7	3.3	0.2	2.4	1.3	1.2	3.7	2.5	0.0	0.4	1.5	1.4	3.5	2.6	2.1	1.1
15	1.3	3.6	3.7	3.2	0.1	0.4	1.5	1.4	3.5	2.6	2.1	0.3	1.7	1.6	3.3	0.2	2.2
16	0.6	1.5	1.4	3.5	2.5	0.0	0.3	1.7	1.6	3.3	0.2	2.2	1.1	0.7	2.7	3.1	0.1

$\langle \mid -w \mid \overline{17} \mid \overset{+}{1-6i}$

r_1 \ r_2	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
0	0.0	3.7	1.2	0.5	2.7	1.5	3.8	2.8	0.2	2.2	0.8	1.8	3.5	0.7	2.5	3.2	1.7
1	3.6	2.1	3.4	1.5	0.4	2.8	1.8	3.5	2.9	0.1	2.3	0.7	0.6	3.2	1.2	2.6	2.7
2	2.8	3.3	2.2	3.1	1.8	1.1	2.9	0.6	3.2	3.9	0.0	3.7	1.2	0.5	2.7	1.5	0.4
3	1.1	2.9	1.3	2.3	2.4	0.6	1.4	3.9	0.5	2.7	3.6	2.1	3.4	1.5	0.4	0.3	1.8
4	0.6	1.4	3.9	1.6	3.7	2.5	0.5	1.7	3.6	0.4	2.8	3.3	2.2	3.1	3.8	[illegible]	0.2
5	0.1	0.5	1.7	3.6	1.9	3.4	2.6	0.4	0.3	3.3	1.1	2.9	1.3	0.8	2.4	3.5	1.4
6	1.7	0.0	0.4	0.3	3.3	0.9	3.1	3.8	1.1	0.2	1.3	1.4	0.1	1.6	0.7	2.5	3.2
7	2.7	0.3	2.1	1.1	0.2	1.3	0.8	2.4	3.5	1.4	0.1	2.3	1.7	0.0	1.9	1.2	2.6

Continuation of table.

$< | -w |_{17} |^{+}_{1+6i}$

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
8	3.8	2.8	0.2	2.2	1.4	0.1	1.6	0.7	2.5	3.2	3.9	0.0	3.7	0.3	2.1	0.9	1.5
9	1.8	3.5	2.9	0.1	2.3	1.7	0.0	1.9	1.2	0.5	2.7	3.6	2.1	3.4	0.2	2.2	0.8
10	0.7	0.6	3.2	3.9	0.0	3.7	0.3	2.1	3.4	1.5	0.4	2.8	3.3	2.2	3.1	0.1	2.3
11	3.7	1.2	0.5	2.7	3.6	2.1	3.4	3.3	2.2	3.1	1.8	1.1	2.9	1.3	2.3	2.4	0.0
12	2.1	3.4	1.5	0.4	2.8	3.3	0.9	3.1	1.3	2.3	2.4	0.6	1.4	3.9	1.6	3.7	2.5
13	2.6	2.2	3.1	1.8	1.1	0.2	1.3	0.8	2.4	1.6	3.7	2.5	0.5	1.7	3.6	1.9	3.4
14	3.1	3.8	2.3	2.4	3.5	1.4	0.1	1.6	0.7	2.5	1.9	3.4	2.6	0.4	0.3	3.3	0.9
15	0.8	2.4	3.5	0.7	2.5	3.2	1.7	0.0	1.9	1.2	2.6	0.9	3.1	3.8	1.1	0.2	1.3
16	1.6	0.7	0.6	3.2	1.2	2.6	2.7	0.3	2.1	0.9	1.5	3.8	0.8	2.4	3.5	1.4	0.1

$< | -w | _{\overline{17}} |^{+}_{1-6i}$

$r_1 \backslash r_2$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
0	0.0	0.8	1.7	3.1	3.8	0.1	2.5	1.5	0.7	2.7	3.5	0.5	2.1	1.8	1.1	3.7	2.8
1	0.5	2.1	1.3	1.9	2.4	3.6	0.0	0.3	1.7	0.6	3.8	3.2	0.4	2.2	0.7	1.4	3.5
2	3.2	0.4	2.2	3.3	0.9	2.5	3.4	2.1	0.2	1.9	1.2	3.6	2.6	1.1	2.3	0.6	3.8
3	3.6	2.6	1.1	2.3	2.8	0.8	0.3	3.1	2.2	0.1	0.9	1.5	3.4	2.7	1.4	3.5	1.2
4	1.5	3.4	2.7	1.4	0.5	2.9	1.3	0.2	2.4	2.3	0.0	0.8	1.7	3.1	0.6	1.6	3.2
5	2.6	1.7	3.1	3.8	1.6	0.4	3.9	3.3	0.1	2.5	0.5	2.1	1.3	0.2	2.4	1.2	1.8
6	0.7	2.7	1.9	2.4	3.6	1.8	1.1	3.7	2.8	0.0	0.3	0.4	3.9	3.3	0.1	2.5	1.5
7	1.7	0.6	3.8	0.9	2.5	3.4	0.7	1.4	3.5	2.9	2.1	1.3	1.1	3.7	2.8	0.0	0.3
8	0.2	1.9	1.2	3.6	0.8	0.3	3.1	0.6	1.6	3.2	0.4	2.2	3.3	1.4	3.5	2.9	2.1
9	2.2	0.1	0.9	1.5	3.4	1.3	0.2	2.4	1.2	3.6	2.6	1.1	2.4	2.8	1.6	3.2	3.9
10	3.7	2.3	0.0	0.8	1.7	3.1	3.3	0.1	0.9	1.5	3.4	2.7	1.4	0.5	2.9	1.8	2.6
11	2.7	3.5	0.5	2.1	1.3	1.9	2.4	2.3	0.0	0.8	1.7	3.1	3.8	1.6	0.4	3.9	0.7
12	0.6	3.8	3.2	0.4	2.2	3.3	0.1	2.5	0.5	2.1	1.3	1.9	2.4	3.6	1.8	1.1	3.7
13	3.5	1.2	3.6	2.6	1.1	3.7	2.8	0.0	0.3	0.4	2.2	3.3	0.9	2.5	3.4	0.7	1.4
14	1.6	3.2	1.5	3.4	0.7	1.4	3.5	2.9	2.1	0.2	1.1	2.3	2.8	0.8	0.3	3.1	0.6
15	1.2	1.8	2.6	0.3	3.1	0.6	1.6	3.2	3.9	2.2	0.1	1.4	0.5	2.9	1.3	0.2	2.4
16	2.5	1.5	3.4	2.7	0.2	2.4	1.2	1.8	2.6	3.7	2.3	0.0	1.6	0.4	3.9	3.3	0.1

$< | -w | _{\overline{17}} |^{+}_{4+5i}$

$r_1 \backslash r_2$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
0	0.0	2.10	0.7	3.9	1.10	0.9	0.6	3.6	3.7	1.7	1.6	2.6	2.9	3.10	1.9	2.7	0.10
1	1.5	2.1	0.9	0.6	3.6	3.7	0.8	0.5	3.2	3.3	3.10	1.9	2.7	0.10	3.8	3.9	1.4
2	0.4	1.8	2.2	0.8	0.5	3.2	3.3	1.2	1.1	2.8	2.10	3.8	3.9	1.4	1.3	3.5	3.4
3	0.7	0.3	1.10	2.3	1.2	1.1	2.8	2.10	1.6	1.5	2.9	0.9	3.5	3.6	0.4	1.8	3.1
4	2.5	0.6	0.2	3.7	2.4	1.6	1.5	2.9	0.9	1.9	1.8	0.10	0.8	3.1	0.7	0.3	1.10
5	3.7	2.6	0.5	0.1	3.3	3.4	1.9	1.8	0.10	0.8	3.9	1.10	1.3	2.1	2.5	0.6	0.2
6	0.1	3.3	2.7	1.1	0.0	2.10	0.7	3.9	1.10	1.3	1.2	3.6	0.4	1.7	2.2	2.6	0.5
7	1.1	0.0	2.10	1.4	1.5	2.1	0.9	0.6	3.6	3.7	1.7	2.2	3.2	0.3	3.10	2.3	2.7
8	1.4	1.5	2.1	0.9	0.4	1.8	2.2	0.8	0.5	3.2	0.3	3.10	2.3	2.8	0.2	3.8	2.4
9	3.4	0.4	1.8	2.2	0.8	0.3	1.10	2.3	1.2	2.5	2.8	0.2	3.8	2.4	2.9	0.1	3.5
10	3.1	0.7	0.3	1.10	2.3	1.2	0.2	3.7	1.7	1.6	2.6	2.9	0.1	3.5	3.4	0.10	0.0
11	2.1	2.5	0.6	0.2	3.7	2.4	1.6	3.2	3.3	3.10	1.9	2.7	0.10	0.0	3.1	0.7	1.3
12	1.7	2.2	2.6	0.5	0.1	3.3	3.10	1.9	2.8	2.10	3.8	3.9	1.4	1.3	2.1	2.5	0.6
13	0.5	3.10	2.3	2.7	1.1	2.8	2.10	3.8	3.9	2.9	0.9	3.5	3.6	0.4	1.7	2.2	2.6
14	2.7	1.1	3.8	2.4	1.6	1.5	2.9	0.9	3.5	3.6	0.10	0.8	3.1	3.2	0.3	3.10	2.3
15	2.4	1.4	1.5	3.3	3.4	1.9	1.8	0.10	0.8	3.1	3.2	1.3	1.2	2.5	2.8	0.2	3.8
16	3.5	3.4	1.9	1.8	2.10	0.7	3.9	1.10	1.3	1.2	2.5	2.8	1.7	1.6	2.6	2.9	0.1

$< \mid -w \mid \overline{17} \mid \overset{\rightarrow}{4-5i}$

$r_1 \backslash r_2$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
0	0.0	2.5	0.4	3.10	1.5	0.10	0.1	3.1	3.4	1.4	1.1	2.1	2.10	3.5	1.10	2.4	0.5
1	1.7	2.1	2.6	0.3	3.8	1.8	1.3	0.0	2.5	0.4	3.10	1.5	2.2	0.7	3.1	3.4	1.4
2	3.10	3.9	2.2	2.7	0.2	3.5	1.10	1.7	2.1	2.6	0.3	3.8	1.8	2.3	0.6	2.5	2.9
3	0.9	3.8	3.6	2.3	0.10	0.1	3.1	3.4	3.9	2.2	2.7	0.2	3.5	1.10	2.4	1.3	2.6
4	2.7	0.8	3.5	3.2	2.4	1.3	0.0	2.5	0.4	3.6	2.3	0.10	0.1	3.1	2.8	1.4	1.7
5	3.9	0.10	1.2	3.1	2.8	1.4	1.7	2.1	2.6	0.3	3.2	2.4	1.3	1.6	2.5	2.9	3.10
6	3.8	3.6	1.3	1.6	2.5	2.9	3.10	3.9	2.2	2.7	0.2	2.8	0.7	1.7	1.9	2.6	0.9
7	0.8	3.5	3.2	1.7	1.9	2.6	0.9	3.8	3.6	2.3	0.10	1.2	2.9	0.6	3.9	3.7	2.7
8	0.10	1.2	3.1	2.8	3.9	3.7	2.7	0.8	3.5	3.2	2.10	1.3	1.6	0.9	0.5	3.6	3.3
9	2.10	1.3	1.6	2.5	2.9	3.6	3.3	0.10	1.2	1.5	2.8	0.7	1.7	1.9	0.8	1.1	3.2
10	2.3	0.7	1.7	1.9	2.6	0.9	3.2	2.10	0.3	1.6	1.8	2.9	0.6	3.9	3.7	1.2	1.5
11	1.8	2.9	0.6	3.9	3.7	2.7	0.8	2.2	0.7	0.2	1.9	1.10	0.9	0.5	3.6	3.3	1.6
12	1.9	1.10	0.9	0.5	3.6	3.3	0.4	1.2	2.3	0.6	0.1	3.7	3.4	0.8	1.1	3.2	2.10
13	0.7	3.7	3.4	0.8	1.1	2.1	2.10	0.3	1.6	2.4	0.5	0.0	3.3	0.4	1.2	1.5	2.8
14	2.9	0.6	3.3	0.4	3.10	1.5	2.2	0.7	0.2	1.9	1.4	1.1	2.1	2.10	0.3	1.6	1.8
15	1.10	0.9	0.5	2.6	0.3	3.8	1.8	2.3	0.6	0.1	3.7	3.10	1.5	2.2	0.7	0.2	1.9
16	3.7	3.4	1.4	1.1	2.7	0.2	3.5	1.10	2.4	0.5	0.0	3.3	2.8	1.8	2.3	0.6	0.1

$< \mid -w \mid \overline{25} \mid \overset{\rightarrow}{2-5i}$

$r_1 \backslash r_2$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24
0	0.0	1.4	0.2	0.6	3.6	3.5	1.3	2.4	0.3	2.7	0.5	3.2	1.7	3.7	1.2	2.5	0.7	2.3	0.4	3.3	1.5	1.6	2.6	2.2	3.4
1	0.5	2.1	1.7	0.1	1.2	3.3	0.7	1.6	2.5	1.1	1.5	0.4	2.6	0.2	3.4	1.4	3.5	0.6	2.4	0.3	3.1	0.5	3.6	2.7	3.7
2	3.4	0.4	2.2	0.2	0.0	1.4	3.1	0.6	3.6	3.6	1.3	0.5	0.3	2.7	0.1	3.2	1.7	0.7	1.2	2.5	1.1	2.3	0.4	3.3	1.1
3	1.3	3.2	0.3	3.7	0.1	2.1	1.7	2.3	1.2	3.3	0.7	1.6	0.4	1.1	1.5	0.0	2.6	0.2	0.6	1.4	3.5	1.3	2.4	3.6	3.1
4	2.3	1.6	2.6	1.1	3.4	0.0	2.2	0.2	2.4	1.4	3.1	0.6	3.6	0.3	1.3	0.5	2.1	2.7	0.1	1.2	1.7	0.7	2.3	2.5	3.3
5	3.1	2.4	3.6	2.7	1.3	3.2	2.1	3.7	0.1	2.5	1.7	2.3	1.2	3.3	1.1	1.6	0.4	2.2	1.5	0.0	1.4	2.2	0.6	2.4	3.5
6	0.7	2.3	2.5	3.3	1.5	1.6	2.6	2.2	3.4	0.0	3.5	0.2	2.4	1.4	3.1	1.3	3.6	0.3	3.7	0.5	3.2	1.7	3.7	1.2	2.5
7	3.5	0.6	2.4	3.5	3.1	0.5	3.6	2.7	3.7	3.2	2.1	0.7	0.1	2.5	1.7	2.3	1.6	3.3	1.1	1.5	0.4	2.6	0.2	3.4	1.4
8	1.7	0.7	1.2	2.5	0.7	2.3	0.4	3.3	1.5	3.4	2.6	2.2	0.6	0.0	3.5	0.2	2.4	3.6	3.1	1.3	0.5	0.3	2.7	0.1	3.2
9	2.6	0.2	0.6	1.4	3.5	0.6	2.4	0.3	3.1	0.5	3.2	2.7	3.7	1.2	2.1	0.7	0.1	1.2	3.3	0.7	1.6	0.4	1.1	1.5	0.0
10	2.1	2.7	0.1	1.2	1.7	0.7	1.2	2.5	1.1	2.3	0.4	2.6	1.5	3.4	1.4	2.2	0.2	0.0	1.4	3.1	0.6	3.6	0.3	1.3	0.5
11	0.4	2.2	1.5	0.0	1.4	0.2	0.6	1.4	3.5	1.3	2.4	0.3	2.7	0.5	3.2	2.1	3.7	0.1	2.1	1.7	2.3	1.2	3.3	1.1	1.6
12	3.6	0.3	3.7	0.5	2.1	1.7	0.1	1.2	1.7	0.7	1.6	2.5	1.1	1.5	1.6	2.6	2.2	3.4	0.0	2.2	0.2	2.4	1.4	3.1	1.3
13	1.6	3.3	1.1	3.4	0.4	2.2	0.2	0.0	1.4	0.2	0.6	3.6	3.5	3.1	0.5	3.6	2.7	3.7	3.2	2.1	3.7	0.1	2.5	1.7	2.3
14	2.4	3.6	3.1	1.3	3.2	0.3	3.7	0.1	2.1	1.7	0.1	1.2	2.5	0.7	2.3	0.4	3.3	1.5	3.4	2.6	2.2	3.4	0.0	3.5	0.2
15	0.1	2.5	3.3	2.3	1.6	2.6	1.1	3.4	0.0	2.2	0.2	3.4	1.4	3.5	0.6	2.4	0.3	3.1	0.5	3.2	2.7	3.7	3.2	2.1	0.7
16	0.6	0.0	3.5	3.1	2.4	3.6	2.7	1.3	3.2	2.1	2.7	0.1	3.2	1.7	0.7	1.2	2.5	1.1	2.3	0.4	2.6	1.5	3.4	2.6	2.2
17	3.7	1.2	2.1	0.7	2.3	2.5	3.3	1.5	1.6	0.4	2.2	1.5	0.0	2.6	0.2	0.6	1.4	3.5	1.3	2.4	0.3	2.7	0.5	3.2	2.7
18	1.5	3.4	1.4	2.2	0.6	2.4	3.5	3.1	0.5	3.6	0.3	3.7	0.5	2.1	2.7	0.1	1.2	1.7	0.7	1.6	2.5	1.1	1.5	0.4	2.6
19	2.7	0.5	3.2	1.7	3.7	1.2	2.5	1.7	2.3	1.6	3.3	1.1	3.4	0.4	2.2	1.5	0.0	1.4	0.2	0.6	3.6	3.5	1.3	0.5	0.3
20	1.1	1.5	0.4	2.6	0.2	3.4	0.0	3.5	0.2	2.4	3.6	3.1	1.3	3.2	0.3	3.7	0.5	2.1	1.7	0.1	1.2	3.3	0.7	1.6	0.4
21	0.3	1.3	0.5	0.3	2.7	3.7	3.2	2.1	0.7	0.1	2.5	3.3	2.3	1.6	2.6	1.1	3.4	0.4	2.2	0.2	0.0	1.4	3.1	0.6	3.6
22	3.3	1.1	1.6	0.4	3.3	1.5	3.4	2.6	2.2	0.6	0.0	3.5	3.1	2.4	3.6	2.7	1.3	3.2	0.3	3.7	0.1	2.1	1.7	2.3	1.2
23	1.4	3.1	1.3	2.4	0.3	3.1	0.5	3.2	2.7	3.7	1.2	2.1	0.7	2.3	2.5	3.3	1.5	1.6	2.6	1.1	3.4	0.0	2.2	0.2	2.4
24	2.5	1.7	0.7	1.6	2.5	1.1	2.3	0.4	2.6	1.5	3.4	1.4	2.2	0.6	2.4	3.5	3.1	0.5	3.6	2.7	1.3	3.2	2.1	3.7	0.1

< | —w | $\bar{25}$ | $\overset{+}{2-6i}$

r_1 \ r_2	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24
0	0.0	3.7	0.5	2.3	1.3	1.1	3.6	0.7	2.6	1.4	2.1	3.5	0.4	2.4	1.5	0.1	3.4	0.6	2.7	1.6	3.1	3.3	0.3	2.5	1.7
1	2.5	2.1	3.5	1.2	2.4	1.5	1.3	3.4	3.6	0.2	1.6	2.2	3.3	0.3	0.4	1.7	0.0	3.2	0.5	3.7	2.7	2.3	3.1	1.1	2.6
2	0.2	2.6	2.2	3.3	1.4	0.4	1.7	1.5	3.2	3.4	0.1	2.7	0.6	3.1	1.1	0.3	0.7	2.1	2.5	1.2	3.5	3.7	2.4	2.3	3.6
3	3.4	0.1	0.2	0.6	3.1	1.6	0.3	2.7	1.7	2.5	3.2	0.0	3.7	0.5	2.3	1.3	1.1	3.6	2.2	2.6	1.4	3.3	3.5	1.5	2.4
4	0.4	3.2	0.0	0.1	0.5	2.3	2.7	1.1	3.6	0.7	2.6	2.5	2.1	3.5	1.2	2.4	1.5	1.3	3.4	0.6	0.2	1.6	0.3	3.3	1.7
5	0.7	0.3	2.5	2.1	0.0	1.2	2.4	3.7	1.3	3.4	3.6	0.2	2.6	2.2	3.3	1.4	0.4	1.7	1.5	3.2	0.5	0.6	2.7	1.1	3.1
6	2.3	3.6	1.1	2.6	2.2	2.1	1.4	0.4	3.5	1.5	3.2	3.4	0.1	0.2	0.6	3.1	1.6	0.3	0.7	1.7	0.0	1.2	0.5	3.7	1.3
7	1.5	2.4	3.4	1.3	0.2	0.6	2.2	1.6	0.3	3.3	1.7	2.5	3.2	0.0	0.1	0.5	2.3	2.7	1.1	2.6	0.7	2.1	1.4	1.2	3.5
8	3.3	1.7	0.4	3.2	1.5	0.1	0.5	0.6	2.7	1.1	3.1	0.7	2.6	2.5	2.1	0.0	1.2	2.4	3.7	1.3	0.2	3.6	2.2	1.6	1.4
9	1.6	3.1	0.7	0.3	2.5	1.7	0.0	1.2	0.5	3.7	1.3	2.3	3.6	0.2	2.6	2.2	2.1	3.5	0.4	2.4	1.5	0.1	3.4	0.6	2.7
10	3.7	2.7	2.3	3.6	1.1	2.6	0.7	2.1	1.4	1.2	3.5	1.5	2.4	3.4	0.1	0.2	1.6	2.2	3.3	0.3	0.4	1.7	0.0	3.2	0.5
11	1.2	3.5	3.7	2.4	3.4	1.3	0.2	3.6	2.2	1.6	1.4	3.3	1.7	0.4	3.2	0.5	0.1	2.7	0.6	3.1	1.1	0.3	0.7	2.1	2.5
12	2.6	1.4	3.3	3.5	0.4	3.2	1.5	0.1	3.4	0.6	2.7	1.6	3.1	0.7	1.7	2.5	1.2	0.0	3.7	0.5	2.3	1.3	1.1	3.6	2.2
13	0.6	0.2	1.6	3.1	3.3	0.3	2.5	1.7	0.0	3.2	0.5	3.7	2.7	1.1	3.6	0.7	2.6	1.4	2.1	3.5	1.2	2.4	1.5	1.3	3.4
14	3.2	0.5	0.1	2.7	2.3	3.1	1.1	2.6	0.7	2.1	2.5	1.2	2.4	3.7	1.3	3.4	3.6	0.2	1.6	2.2	3.3	1.4	0.4	1.7	1.5
15	1.7	2.5	1.2	0.0	3.7	2.4	2.3	1.3	0.2	1.6	2.2	2.1	1.4	0.4	3.5	1.5	3.2	3.4	0.1	2.7	0.6	3.1	1.6	0.3	0.7
16	3.6	0.7	2.6	1.4	2.1	3.5	0.4	2.4	1.5	0.1	0.2	0.6	2.2	1.6	0.3	3.3	1.7	2.5	3.2	0.0	3.7	0.5	2.3	2.7	1.1
17	1.3	3.4	3.6	0.2	1.6	2.2	3.3	0.3	0.4	3.2	0.0	0.1	0.5	0.6	2.7	1.1	3.1	0.7	2.6	2.5	2.1	3.5	1.2	2.4	3.7
18	3.5	1.5	3.2	3.4	0.1	2.7	0.6	3.1	1.1	0.3	2.5	2.1	0.0	1.2	0.5	3.7	1.3	2.3	3.6	0.2	2.6	2.2	3.3	1.4	0.4
19	0.3	3.3	1.7	2.5	3.2	0.0	3.7	2.7	2.3	3.6	1.1	2.6	2.2	2.1	1.4	1.2	3.5	1.5	2.4	3.4	0.1	0.2	0.6	3.1	1.6
20	2.7	1.1	3.1	0.7	2.6	2.5	1.2	3.5	3.7	2.4	3.4	1.3	0.2	0.6	2.2	1.6	1.4	3.3	1.7	0.4	3.2	0.0	0.1	0.5	2.3
21	2.4	3.7	1.3	2.3	3.6	2.2	2.6	1.4	3.3	3.5	0.4	3.2	1.5	0.1	0.5	0.6	2.7	1.6	3.1	0.7	0.3	2.5	2.1	0.0	1.2
22	1.4	0.4	3.5	1.5	1.3	3.4	0.6	0.2	1.6	3.1	3.3	0.3	2.5	1.7	0.0	1.2	0.5	3.7	2.7	2.3	3.6	1.1	2.6	2.2	2.1
23	2.2	1.6	0.3	0.4	1.7	1.5	3.2	0.5	0.1	2.7	2.3	3.1	1.1	2.6	0.7	2.1	1.4	1.2	3.5	3.7	2.4	3.4	1.3	0.2	0.6
24	0.5	0.6	3.1	1.1	0.3	0.7	1.7	2.5	1.2	0.0	3.7	2.4	2.3	1.3	0.2	3.6	2.2	1.6	1.4	3.3	3.5	0.4	3.2	1.5	0.1

< | —w | $\bar{25}$ | $\overset{+}{1-6i}$

r_1 \ r_2	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24
0	0.0	3.4	1.7	2.1	1.1	0.3	1.8	0.4	0.8	0.6	1.3	0.7	3.6	1.6	2.7	3.3	2.6	2.8	2.4	3.8	2.3	3.1	0.1	3.7	1.4
1	2.5	2.1	3.1	0.8	2.2	1.4	0.2	0.6	1.1	0.7	0.5	1.6	1.2	3.3	1.9	2.8	1.3	3.8	2.9	2.5	3.5	3.7	2.4	0.0	3.4
2	3.1	2.6	2.2	2.4	0.2	2.3	1.7	0.1	0.5	1.4	1.2	0.4	1.9	1.5	1.3	0.9	2.9	1.6	3.5	3.9	2.6	3.2	3.4	2.5	2.7
3	2.8	2.4	3.8	2.3	2.5	0.1	3.7	0.3	0.0	0.4	1.7	1.5	1.1	0.9	1.8	1.6	0.8	3.9	1.9	3.2	3.6	3.8	2.7	3.3	2.6
4	3.8	2.9	2.5	3.5	3.7	2.6	0.0	3.4	0.2	2.1	1.1	0.3	1.8	1.4	0.8	0.6	1.9	0.7	3.6	0.9	2.7	3.3	0.8	2.8	1.3
5	1.6	3.5	3.9	2.6	3.2	3.4	3.8	2.1	3.1	0.1	2.2	1.4	0.2	0.6	1.7	0.7	0.5	0.9	1.2	3.3	0.8	1.5	1.3	0.7	2.9
6	3.9	1.9	3.2	3.6	3.8	2.7	3.1	3.5	2.2	2.4	0.0	2.3	1.7	0.1	0.5	0.3	1.2	0.4	0.8	1.5	1.1	0.7	1.8	1.6	1.2
7	1.5	3.6	0.9	2.7	3.3	3.5	2.8	2.4	3.2	2.3	2.5	2.1	3.7	0.3	0.0	0.4	0.2	1.5	1.1	0.1	1.8	1.4	1.2	0.6	1.9
8	0.9	1.8	3.3	0.8	2.8	2.3	3.2	2.9	2.5	2.7	3.7	2.6	2.2	3.4	0.2	2.1	1.1	0.1	1.8	1.4	0.0	0.6	1.7	1.5	0.5
9	0.4	0.8	0.6	1.3	0.7	2.9	1.6	2.7	3.9	2.6	2.8	3.4	3.8	2.3	3.1	0.1	2.2	2.4	0.0	2.3	1.7	2.1	0.5	0.3	1.8
10	0.6	1.1	0.7	0.5	1.6	1.2	3.9	1.9	2.8	3.6	3.8	2.9	3.1	3.5	3.7	2.4	3.2	2.3	2.5	2.1	3.7	0.3	2.2	0.4	0.2
11	0.1	0.5	1.4	1.2	0.4	1.9	1.5	3.6	0.0	2.9	3.3	3.5	3.9	2.4	3.2	3.6	2.5	2.7	3.7	2.6	2.2	3.4	0.2	2.3	1.1
12	1.4	0.0	0.4	1.7	1.5	1.1	0.9	1.8	3.3	0.8	3.9	1.3	3.2	3.6	1.6	2.7	3.3	2.6	2.8	3.4	3.8	2.3	3.1	0.1	3.7
13	3.4	1.7	2.1	1.1	0.3	1.8	1.4	0.8	0.6	1.3	0.7	3.6	1.6	1.2	3.3	1.9	2.8	1.3	3.8	2.9	3.1	3.5	3.7	2.4	0.0
14	2.1	3.1	0.3	2.2	1.4	0.2	0.6	1.7	0.7	0.5	1.6	1.2	0.4	1.9	1.5	1.3	0.9	2.9	1.6	3.5	3.9	2.4	3.2	3.4	2.5
15	2.6	2.2	2.4	[illegible]	2.3	1.7	0.1	0.5	0.3	1.2	0.4	1.7	1.5	1.1	0.9	1.8	1.6	0.8	3.9	1.9	3.2	3.6	2.5	2.7	3.1
16	2.4	3.8	[illegible]	[illegible]	[illegible]	3.7	0.3	0.0	0.4	0.2	2.1	1.1	0.3	1.8	1.4	0.8	0.6	1.9	0.7	3.6	0.9	2.7	3.3	2.6	2.8
17	2.9	2.5	[illegible]	[illegible]	[illegible]	0.0	3.4	0.2	2.1	3.1	0.1	2.2	1.4	0.2	0.6	1.7	0.7	0.5	0.9	1.2	3.3	0.8	2.8	1.3	3.8
18	3.5	3.9	[illegible]	[illegible]	3.4	3.8	2.1	3.1	0.1	2.2	2.4	0.0	2.3	1.7	0.1	0.5	0.3	1.2	0.4	0.8	1.5	1.3	0.7	2.9	1.6
19	1.9	3.2	3.6	3.8	2.7	3.1	3.5	2.8	2.4	3.2	2.3	2.5	2.1	3.7	0.3	0.0	0.4	0.2	1.5	1.1	0.7	1.8	1.6	1.2	3.9
20	3.6	0.9	2.7	3.3	3.5	2.8	1.3	3.2	2.9	2.5	2.7	3.7	2.6	2.2	3.4	0.2	2.1	1.1	0.1	1.8	1.4	1.2	0.6	1.9	1.5
21	1.8	3.3	0.8	2.8	1.3	0.7	2.9	1.6	2.7	3.9	2.6	2.8	3.4	3.8	2.3	3.1	0.1	2.2	1.4	0.0	0.6	1.7	1.5	0.5	0.9
22	0.8	0.6	1.3	0.7	1.8	1.6	1.2	3.9	1.9	2.8	3.6	3.8	2.9	3.1	3.5	3.7	2.4	0.0	2.3	1.7	2.1	0.5	0.3	1.8	0.4
22	1.1	0.7	0.5	1.4	1.2	0.6	1.9	1.5	3.6	0.9	2.9	3.3	3.5	3.9	2.4	3.2	3.4	2.5	2.1	3.7	0.3	2.2	0.4	0.2	0.6
24	0.5	1.4	0.0	0.4	1.7	1.5	0.5	0.9	1.8	3.3	0.8	3.9	1.3	3.2	3.6	2.5	2.7	3.1	2.6	2.2	3.4	0.2	2.3	1.1	0.1

$< \mid -w \mid \overline{25} \mid \overset{+}{2-5t}$

r_1 \ r_2	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24
0	0.0	3.7	0.5	2.3	1.3	1.1	3.6	0.7	2.6	1.4	2.1	3.5	0.4	2.4	1.5	0.1	3.4	0.6	2.7	1.6	3.1	3.3	0.3	2.5	1.7
1	2.5	2.1	3.5	1.2	2.4	1.5	1.3	3.4	3.6	0.2	1.6	2.2	3.3	0.3	0.4	1.7	0.0	3.2	0.5	3.7	2.7	2.3	3.1	1.1	2.6
2	0.2	2.6	2.2	3.3	1.4	0.4	1.7	1.5	3.2	3.4	0.1	2.7	0.6	3.1	1.1	0.3	0.7	2.1	2.5	1.2	3.5	3.7	2.4	2.3	3.6
3	3.4	0.1	0.2	0.6	3.1	1.6	0.3	2.7	1.7	2.5	3.2	0.0	3.7	0.5	2.3	1.3	1.1	3.6	2.2	2.6	1.4	3.3	3.5	1.5	2.4
4	0.4	3.2	0.0	0.1	0.5	2.3	2.7	1.1	3.6	0.7	2.6	2.5	2.1	3.5	1.2	2.4	1.5	1.3	3.4	0.6	0.2	1.6	0.3	3.3	1.7
5	0.7	0.3	2.5	2.1	0.0	1.2	2.4	3.7	1.3	3.4	3.6	0.2	2.6	2.2	3.3	1.4	0.4	1.7	1.5	3.2	0.5	0.6	2.7	1.1	3.1
6	2.3	3.6	1.1	2.6	2.2	2.1	1.4	0.4	3.5	1.5	3.2	3.4	0.1	0.2	0.6	3.1	1.6	0.3	0.7	1.7	0.0	1.2	0.5	3.7	1.3
7	1.5	2.4	3.4	1.3	0.2	0.6	2.2	1.6	0.3	3.3	1.7	2.5	3.2	0.0	0.1	0.5	2.3	2.7	1.1	2.6	0.7	2.1	1.4	1.2	3.5
8	3.3	1.7	0.4	3.2	1.5	0.1	0.5	0.6	2.7	1.1	3.1	0.7	2.6	2.5	2.1	0.0	1.2	2.4	3.7	1.3	0.2	3.6	2.2	1.6	1.4
9	1.6	3.1	0.7	0.3	2.5	1.7	0.0	1.2	0.5	3.7	1.3	2.3	3.6	0.2	2.6	2.2	2.1	3.5	0.4	2.4	1.5	0.1	3.4	0.6	2.7
10	3.7	2.7	2.3	3.6	1.1	2.6	0.7	2.1	1.4	1.2	3.5	1.5	2.4	3.4	0.1	0.2	1.6	2.2	3.3	0.3	0.4	1.7	0.0	3.2	0.5
11	1.2	3.5	3.7	2.4	3.4	1.3	0.2	3.6	2.2	1.6	1.4	3.3	1.7	0.4	3.2	0.5	0.1	2.7	0.6	3.1	1.1	0.3	0.7	2.1	2.5
12	2.6	1.4	3.3	3.5	0.4	3.2	1.5	0.1	3.4	0.6	2.7	1.6	3.1	0.7	1.7	2.5	1.2	0.0	3.7	0.5	2.3	1.3	1.1	3.6	2.2
13	0.6	0.2	1.6	3.1	3.3	0.3	2.5	1.7	0.0	3.2	0.5	3.7	2.7	1.1	3.6	0.7	2.6	1.4	2.1	3.5	1.2	2.4	1.5	1.3	3.4
14	3.2	0.5	0.1	2.7	2.3	3.1	1.1	2.6	0.7	2.1	2.5	1.2	2.4	3.7	1.3	3.4	3.6	0.2	1.6	2.2	3.3	1.4	0.4	1.7	1.5
15	1.7	2.5	1.2	0.0	3.7	2.4	2.3	1.3	0.2	1.6	2.2	2.1	1.4	0.4	3.5	1.5	3.2	3.4	0.1	2.7	0.6	3.1	1.6	0.3	0.7
16	3.6	0.7	2.6	1.4	2.1	3.5	0.4	2.4	1.5	0.1	0.2	0.6	2.2	1.6	0.3	3.3	1.7	2.5	3.2	0.0	3.7	0.5	2.3	2.7	1.1
17	1.3	3.4	3.6	0.2	1.6	2.2	3.3	0.3	0.4	3.2	0.0	0.1	0.5	0.6	2.7	1.1	3.1	0.7	2.6	2.5	2.1	3.5	1.2	2.4	3.7
18	3.5	1.5	3.2	3.4	0.1	2.7	0.6	3.1	1.1	0.3	2.5	2.1	0.0	1.2	0.5	3.7	1.3	2.3	3.6	0.2	2.6	2.2	3.3	1.4	0.4
19	0.3	3.3	1.7	2.5	3.2	0.0	3.7	2.7	2.3	3.6	1.1	2.6	2.2	2.1	1.4	1.2	3.5	1.5	2.4	3.4	0.1	0.2	0.6	3.1	1.6
20	2.7	1.1	3.1	0.7	2.6	2.5	1.2	3.5	3.7	2.4	3.4	1.3	0.2	0.6	2.2	1.6	1.4	3.3	1.7	0.4	3.2	0.0	0.1	0.5	2.3
21	2.4	3.7	1.3	2.3	3.6	2.2	2.6	1.4	3.3	3.5	0.4	3.2	1.5	0.1	0.5	0.6	2.7	1.6	3.1	0.7	0.3	2.5	2.1	0.0	1.2
22	1.4	0.4	3.5	1.5	1.3	3.4	0.6	0.2	1.6	3.1	3.3	0.3	2.5	1.7	0.0	1.2	0.5	3.7	2.7	2.3	3.6	1.1	2.6	2.2	2.1
23	2.2	1.6	0.3	0.4	1.7	1.5	3.2	0.5	0.1	2.7	2.3	3.1	1.1	2.6	0.7	2.1	1.4	1.2	3.5	3.7	2.4	3.4	1.3	0.2	0.6
24	0.5	0.6	3.1	1.1	0.3	0.7	1.7	2.5	1.2	0.0	3.7	2.4	2.3	1.3	0.2	3.6	2.2	1.6	1.4	3.3	3.5	0.4	3.2	1.5	0.1

$< \mid -w \mid \overline{25} \mid \overset{+}{1-6t}$

r_1 \ r_2	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24
0	0.0	3.4	1.7	2.1	1.1	0.3	1.8	0.4	0.8	0.6	1.3	0.7	3.6	1.6	2.7	3.3	2.6	2.8	2.4	3.8	2.3	3.1	0.1	3.7	1.4
1	2.5	2.1	3.1	0.3	2.2	1.4	0.2	0.6	1.1	0.7	0.5	1.6	1.2	3.3	1.9	2.8	1.3	3.8	2.9	2.5	3.5	3.7	2.4	0.0	3.4
2	3.1	2.6	2.2	2.4	0.2	2.3	1.7	0.1	0.5	1.4	1.2	0.4	1.9	1.5	1.3	0.9	2.9	1.6	3.5	3.9	2.6	3.2	3.4	2.5	2.7
3	2.8	2.4	3.8	2.3	2.5	0.1	3.7	0.3	0.0	0.4	1.7	1.5	1.1	0.9	1.8	1.6	0.8	3.9	1.9	3.2	3.6	3.8	2.7	3.3	2.5
4	3.8	2.9	2.5	3.5	3.7	2.6	0.0	3.4	0.2	2.1	1.1	0.3	1.8	1.4	0.8	0.6	1.9	0.7	3.6	0.9	2.7	3.3	0.8	2.8	1.3
5	1.6	3.5	3.9	2.6	3.2	3.4	3.8	2.1	3.1	0.1	2.2	1.4	0.2	0.6	1.7	0.7	0.5	0.9	1.2	3.3	0.8	1.5	1.3	0.7	2.9
6	3.9	1.9	3.2	3.6	3.8	2.7	3.1	3.5	2.2	2.4	0.0	2.3	1.7	0.1	0.5	0.3	1.2	0.4	0.8	1.5	1.1	0.7	1.3	1.6	1.2
7	1.5	3.6	0.9	2.7	3.3	3.5	2.8	2.4	3.2	2.3	2.5	2.1	3.7	0.3	0.0	0.4	0.2	1.5	1.1	0.1	1.8	1.4	1.2	0.6	1.9
8	0.9	1.8	3.3	0.8	2.8	2.3	3.2	2.9	2.5	2.7	3.7	2.6	2.2	3.4	0.2	2.1	1.1	0.1	1.8	1.4	0.0	0.6	1.7	1.5	0.5
9	0.4	0.8	0.6	1.3	0.7	2.9	1.6	2.7	3.9	2.6	2.8	3.4	3.8	2.3	3.1	0.1	2.2	2.4	0.0	2.3	1.7	2.1	0.5	0.3	1.8
10	0.6	1.1	0.7	0.5	1.6	1.2	3.9	1.9	2.8	3.6	3.8	2.9	3.1	3.5	3.7	2.4	3.2	2.3	2.5	2.1	3.7	0.3	2.2	0.4	0.2
11	0.1	0.5	1.4	1.2	0.4	1.9	1.5	3.6	0.0	2.9	3.3	3.5	3.9	2.4	3.2	3.6	2.5	2.7	3.[illegible]	2.6	2.2	3.4	0.2	2.3	1.1
12	1.4	0.0	0.4	1.7	1.5	1.1	0.9	1.8	3.3	0.8	3.9	1.3	3.2	3.6	1.6	2.7	3.3	2.6	2.8	3.4	3.8	2.3	3.1	0.1	3.7
13	3.4	1.7	2.1	1.1	0.3	1.8	1.4	0.8	0.6	1.3	0.7	3.6	1.6	1.2	3.3	1.9	2.8	1.3	3.8	2.9	3.1	3.5	3.7	2.4	0.0
14	2.1	3.1	0.3	2.2	1.4	0.2	0.6	1.7	0.7	0.5	1.6	1.2	0.4	1.9	1.5	1.3	0.9	2.9	1.6	3.5	3.9	2.4	3.2	3.4	2.5
15	2.6	2.2	2.4	0.2	2.3	1.7	0.1	0.5	0.3	1.2	0.4	1.7	1.5	1.1	0.9	1.8	1.6	0.8	3.9	1.9	3.2	3.6	2.5	2.7	3.1
16	2.4	3.8	2.3	2.5	0.1	3.7	0.3	0.0	0.4	0.2	2.1	1.1	0.3	1.8	1.4	0.8	0.6	1.9	0.7	3.6	0.9	2.7	3.3	2.6	2.8
17	2.9	2.5	3.5	3.7	2.6	0.0	3.4	0.2	2.1	3.1	0.1	2.2	1.4	0.2	0.6	1.7	0.7	0.5	0.9	1.2	3.3	0.8	2.8	1.3	3.8
18	3.5	3.9	2.6	3.2	3.4	3.8	2.1	3.1	0.1	2.2	2.4	0.0	2.3	1.7	0.1	0.5	0.3	1.2	0.4	0.8	1.5	1.3	0.7	2.9	1.6
19	1.9	3.2	3.6	3.8	2.7	3.1	3.5	2.8	2.4	3.2	2.3	2.5	2.1	3.7	0.3	0.0	0.4	0.2	1.5	1.1	0.7	1.8	1.6	1.2	3.9
20	3.6	0.9	2.7	3.3	3.5	2.8	1.3	3.2	2.9	2.5	2.7	3.7	2.6	2.2	3.4	0.2	2.1	1.1	0.1	1.8	1.4	1.2	0.6	1.9	1.5
21	1.8	3.3	0.8	2.8	1.3	0.7	2.9	1.6	2.7	3.9	2.6	2.8	3.4	3.8	2.3	3.1	0.1	2.2	1.4	0.0	0.6	1.7	1.5	0.5	0.9
22	0.8	0.6	1.3	0.7	1.8	1.6	1.2	3.9	1.9	2.8	3.6	3.8	2.9	3.1	3.5	3.7	2.4	0.0	2.3	1.7	2.1	0.5	0.3	1.8	0.4
22	1.1	0.7	0.5	1.4	1.2	0.6	1.9	1.5	3.6	0.9	2.9	3.3	3.5	3.9	2.1	3.2	3.4	2.5	2.1	3.7	0.3	2.2	0.4	0.2	0.6
24	0.5	1.4	0.0	0.4	1.7	1.5	0.5	0.9	1.8	3.3	0.8	3.9	1.3	3.2	3.6	2.5	2.7	3.1	2.6	2.2	3.4	0.2	2.3	1.1	0.1

$< \mid -w \mid {}^{-}_{25} \mid {}^{+}_{1-6i}$

r_1 \ r_2	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24
0	0.0	1.6	1.1	1.7	0.7	2.8	0.9	2.6	3.9	3.4	3.8	0.1	2.4	0.4	2.1	1.8	1.4	1.9	0.6	2.9	0.8	2.7	3.7	3.1	3.6
1	0.4	2.1	1.8	1.4	1.9	0.6	2.9	0.8	2.7	3.7	3.1	3.6	0.0	2.5	1.1	2.2	0.7	1.6	0.9	1.2	3.9	1.3	3.9	3.5	2.4
2	2.5	1.1	2.2	0.7	1.6	0.9	1.2	3.9	1.3	3.8	3.5	2.4	3.4	2.1	0.3	1.4	2.3	0.6	1.8	0.8	1.5	3.7	3.3	3.6	0.0
3	2.1	0.3	1.4	2.3	0.6	1.8	0.8	1.5	3.7	3.3	3.6	3.2	2.5	3.1	2.2	0.2	1.6	0.5	1.2	0.7	1.3	1.7	3.5	2.4	3.4
4	3.1	2.2	0.2	1.6	0.5	1.2	0.7	1.3	1.7	3.5	2.8	3.4	2.6	0.3	2.4	2.3	0.1	1.8	0.4	1.5	0.6	3.3	2.7	3.2	2.5
5	0.3	2.4	2.3	0.1	1.8	0.4	1.5	0.6	3.3	1.9	3.2	2.9	3.1	2.7	0.2	2.5	0.5	0.0	0.7	1.1	1.7	3.5	2.8	3.8	2.6
6	2.7	0.2	2.5	0.5	0.0	0.7	1.1	1.7	1.2	2.8	0.9	2.6	3.9	2.4	3.8	0.1	0.3	0.4	2.1	0.6	3.3	1.9	3.2	2.9	3.6
7	3.4	3.8	0.1	0.3	0.4	2.1	0.6	1.4	1.9	1.5	2.9	0.8	2.7	3.7	2.5	3.6	0.0	0.2	1.1	1.7	1.2	2.8	0.9	2.6	3.9
8	3.7	3.1	3.6	0.0	0.2	1.1	2.2	1.2	1.6	0.9	1.7	3.9	1.3	3.8	3.5	0.3	3.4	2.1	0.1	1.4	1.9	1.5	2.9	0.8	2.7
9	3.8	3.5	2.4	3.4	2.1	0.1	1.4	2.3	1.5	1.8	0.8	1.9	3.7	3.3	3.6	3.2	0.2	1.1	2.2	1.2	1.6	0.9	1.7	3.9	1.3
10	3.3	3.6	3.2	2.7	3.1	2.2	0.0	1.6	0.5	1.7	0.7	1.3	0.9	3.5	2.8	3.4	2.1	0.1	1.4	2.3	1.5	1.8	0.8	1.9	3.7
11	3.5	2.8	3.4	2.6	0.3	2.4	2.3	2.1	1.8	0.4	1.9	0.6	3.3	0.8	3.2	2.5	3.1	2.2	0.0	1.6	0.5	1.7	0.7	1.3	0.9
12	0.8	3.2	2.9	3.1	2.7	0.2	2.5	0.5	2.2	0.7	1.1	0.9	1.2	2.8	3.4	2.6	0.3	2.4	2.3	2.1	1.8	0.4	1.9	0.6	3.3
13	2.8	1.3	2.6	3.9	2.4	3.8	0.1	0.3	0.4	2.3	0.6	1.4	0.8	3.2	2.9	3.1	2.7	0.2	2.5	0.5	2.2	0.7	1.1	0.9	1.2
14	1.5	2.9	3.3	2.7	3.7	2.5	3.6	0.0	0.2	1.1	0.5	1.2	2.8	1.3	2.6	3.9	2.4	3.8	0.1	0.3	0.4	2.3	0.6	1.4	0.8
15	1.3	1.7	3.9	2.8	3.8	3.5	0.3	3.4	2.1	0.1	1.4	0.8	1.5	2.9	3.3	2.7	3.7	2.5	3.6	0.0	0.2	1.1	0.5	1.2	1.6
16	1.8	3.3	1.9	3.7	2.9	3.6	3.2	0.2	3.1	2.2	1.2	1.6	1.3	1.7	3.9	2.8	3.8	3.5	0.3	3.4	2.1	0.1	1.4	0.4	1.5
17	1.7	0.7	2.8	0.9	3.5	3.9	3.4	2.6	0.1	1.4	2.3	1.5	1.8	3.3	1.9	3.7	2.9	3.6	3.2	0.2	3.1	2.2	0.0	1.6	1.1
18	1.4	1.9	0.6	2.9	0.8	3.2	3.7	3.1	2.7	0.0	1.6	0.5	1.7	0.7	2.8	0.9	3.5	3.9	3.4	2.6	0.1	2.4	2.3	2.1	1.8
19	0.7	1.6	0.9	1.2	3.9	1.3	2.6	0.1	2.4	2.3	2.1	1.8	0.4	1.9	0.6	2.9	0.8	3.2	3.7	3.1	2.7	0.0	2.5	0.5	2.2
20	2.3	0.6	1.8	0.8	1.5	3.7	3.1	2.7	0.0	2.5	0.5	2.2	0.7	1.1	0.9	1.2	3.9	1.3	2.6	3.5	2.4	3.8	2.1	0.3	0.4
21	1.1	0.5	1.2	0.7	1.3	2.6	3.5	2.4	3.8	2.1	0.3	0.4	2.3	0.6	1.4	0.8	1.5	3.7	3.3	2.7	3.2	2.5	3.6	2.2	0.2
22	0.1	1.4	0.4	1.5	3.7	3.3	2.7	3.2	2.5	3.6	2.2	0.2	1.1	0.5	1.2	1.6	1.3	1.7	3.5	2.8	3.8	2.6	0.3	3.4	2.3
23	0.5	0.0	1.6	1.3	1.7	3.5	2.8	3.8	2.6	0.3	3.4	2.3	0.1	1.4	0.4	1.5	1.8	3.3	1.9	3.2	2.9	3.6	2.7	0.2	3.1
24	2.4	0.4	1.5	1.8	3.3	1.9	3.2	2.9	3.6	2.7	0.2	3.1	0.5	0.0	1.6	1.1	1.7	0.7	2.8	0.9	2.6	3.9	3.4	3.8	0.1

$< \mid -w \mid {}^{-}_{25} \mid {}^{+}_{4-5i}$

r_1 \ r_2	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24
0	0.0	3.8	3.2	1.10	0.10	2.7	0.5	0.8	3.5	1.9	3.7	2.2	0.9	2.9	0.2	1.7	3.9	1.5	2.8	2.5	0.7	2.10	3.10	1.2	1.8
1	1.7	2.1	3.5	2.8	3.7	1.3	1.4	1.1	1.2	3.1	3.9	3.3	2.3	0.8	0.10	0.1	3.10	3.6	1.8	2.9	2.6	0.6	0.9	3.8	1.6
2	1.9	3.10	2.2	3.1	2.9	3.3	1.7	0.4	1.5	1.6	2.5	3.6	2.10	2.4	1.2	1.3	0.0	3.8	3.2	1.10	0.10	2.7	0.5	0.8	0.7
3	0.6	3.9	3.8	2.3	2.5	0.10	2.10	3.10	0.3	1.8	1.9	2.6	3.2	0.9	3.4	1.6	1.7	2.1	3.5	2.8	3.7	1.3	1.4	2.9	1.2
4	1.6	0.5	3.6	3.5	2.4	2.6	1.3	0.9	3.8	0.2	1.10	3.9	2.7	2.8	0.8	0.7	1.9	3.10	2.2	3.1	2.9	3.3	2.3	0.4	0.10
5	1.3	1.9	1.1	3.2	3.1	3.4	2.7	1.7	0.8	3.5	0.1	3.7	3.6	1.4	2.9	1.2	0.6	3.9	3.8	2.3	2.5	2.1	2.10	2.4	0.3
6	0.2	1.7	3.9	1.5	2.8	2.5	0.7	1.4	3.10	1.2	3.1	0.0	3.3	3.2	0.4	0.10	1.6	0.5	3.6	3.5	1.9	2.6	2.2	0.9	3.4
7	0.7	0.1	3.10	3.6	1.8	2.9	2.6	0.6	0.4	3.8	1.6	2.5	2.1	2.10	2.8	0.3	1.3	1.9	1.1	0.2	3.1	3.9	2.7	2.3	0.8
8	1.2	0.6	0.0	3.8	3.2	1.10	0.10	2.7	0.5	0.3	3.5	1.9	2.6	2.2	0.9	2.9	0.2	1.7	3.9	1.5	0.1	2.5	3.6	1.4	2.4
9	3.4	1.6	0.5	2.1	3.5	2.8	3.7	1.3	1.4	1.1	0.2	3.1	3.9	2.7	2.3	0.8	0.10	2.10	3.10	0.3	1.8	0.0	2.6	3.2	0.4
10	0.3	0.7	1.9	1.1	2.2	3.1	2.9	3.3	1.7	0.4	1.5	0.1	2.5	3.6	1.4	2.4	2.6	1.3	0.9	3.8	0.2	1.10	2.1	2.7	2.8
11	2.9	0.2	0.6	3.9	1.5	2.3	2.5	0.10	2.10	3.10	0.3	1.8	0.0	2.6	3.2	1.10	3.4	2.7	1.7	0.8	3.5	0.1	3.7	2.2	1.4
12	0.4	0.10	0.1	0.5	3.6	1.8	2.4	2.6	1.3	0.9	3.8	0.2	1.10	2.1	1.5	2.8	3.7	0.7	1.4	3.10	1.2	3.1	0.0	3.3	2.3
13	2.4	0.3	1.3	0.0	1.1	3.2	1.10	3.4	2.7	1.7	0.8	3.5	0.1	3.10	2.2	1.8	2.9	3.3	0.6	0.4	3.8	1.6	2.5	2.1	2.10
14	0.9	3.4	0.2	1.7	2.1	1.5	2.8	3.7	0.7	1.4	3.10	1.2	0.6	0.0	3.8	2.3	1.10	0.10	2.10	0.5	0.3	3.5	1.9	2.6	2.2
15	2.3	0.8	0.7	0.1	3.10	2.2	1.8	2.9	3.3	0.6	0.4	3.4	1.6	0.5	2.1	3.5	2.4	3.7	1.3	0.9	1.1	0.2	3.1	3.9	2.7
16	1.4	2.4	1.2	0.6	0.0	3.8	2.3	1.10	0.10	2.10	2.8	0.3	0.7	1.9	1.1	2.2	3.1	3.4	3.3	1.7	0.8	1.5	0.1	2.5	3.6
17	3.2	0.4	3.4	1.6	0.5	2.1	3.5	2.4	3.7	2.2	0.9	2.9	0.2	0.6	3.9	1.5	2.3	2.5	0.7	2.10	3.10	1.2	1.8	0.0	2.6
18	2.7	2.8	0.3	0.7	1.9	1.1	2.2	3.1	3.4	3.3	2.3	0.8	0.10	0.1	0.5	3.6	1.8	2.4	2.6	0.6	0.9	3.8	1.6	1.10	2.1
19	2.2	1.4	2.9	0.2	0.6	3.9	1.5	1.6	2.5	3.6	2.10	2.4	1.2	1.3	0.0	1.1	3.2	1.10	3.4	2.7	0.5	0.8	3.5	1.9	3.7
20	3.3	2.3	0.4	0.10	0.1	0.5	0.3	1.8	1.9	2.6	3.2	0.9	3.4	1.6	1.7	2.1	1.5	2.8	3.7	0.7	1.4	1.1	1.2	3.1	3.9
21	3.6	2.10	2.4	0.3	1.3	0.9	1.1	0.2	1.10	3.9	2.7	2.8	0.8	0.7	1.9	3.10	2.2	1.8	2.9	3.3	0.6	0.4	1.5	1.6	2.5
22	2.6	3.2	0.9	3.4	3.3	1.7	0.8	1.5	0.1	3.7	3.6	1.4	2.9	1.2	0.6	3.9	3.8	2.3	1.10	0.10	2.10	0.5	0.3	1.8	1.9
23	3.9	2.7	2.8	2.5	0.7	2.10	3.10	1.2	1.8	0.0	3.3	3.2	0.4	0.10	1.6	0.5	3.6	3.5	2.4	3.7	1.3	0.9	1.1	0.2	1.10
24	3.7	3.6	1.8	2.9	2.6	0.6	0.9	3.8	1.6	1.10	2.1	2.10	2.8	0.3	1.3	1.9	1.1	3.2	3.1	3.4	3.3	1.7	0.8	1.5	0.1

$< \,|\, -w \,|\, \overline{25} \,|\, \overset{+}{4-6i}$

$r_1 \backslash r_2$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24
0	0.0	2.10	2.9	3.8	2.8	3.3	0.2	3.10	3.2	1.4	0.3	1.9	0.4	2.4	3.9	2.3	3.4	1.2	1.10	2.2	1.3	0.8	1.8	0.9	0.10
1	0.6	2.1	0.7	0.9	3.5	2.9	2.10	0.1	3.8	2.8	3.10	0.2	3.7	0.3	1.4	3.6	2.4	0.4	1.6	3.4	2.3	1.7	1.2	1.10	0.8
2	1.2	0.5	2.2	0.6	0.8	3.1	0.9	0.7	0.0	3.5	2.9	3.8	0.1	3.3	0.2	3.10	3.2	1.4	0.3	1.9	0.4	2.4	3.9	1.6	2.6
3	2.7	1.6	1.1	2.3	0.5	1.2	2.5	0.8	0.6	2.1	3.1	0.9	3.5	0.0	2.10	0.1	3.8	2.8	3.10	0.2	3.7	0.3	1.4	1.5	1.9
4	3.7	0.10	1.9	1.5	2.4	1.1	1.6	2.6	1.2	0.5	2.2	2.5	0.8	3.1	2.1	0.7	0.0	3.5	2.9	3.8	0.1	3.3	1.3	3.10	1.8
5	1.10	3.3	1.3	3.7	1.8	1.4	1.5	1.9	2.7	1.6	1.1	2.3	2.6	1.2	2.5	2.2	0.6	2.1	3.1	0.9	3.5	3.9	2.10	1.7	3.8
6	3.5	3.4	2.10	1.7	3.3	1.10	3.10	1.8	3.7	0.10	1.9	1.5	2.4	2.7	1.6	2.6	2.3	0.5	2.2	2.5	0.3	3.1	3.6	0.7	2.9
7	3.6	3.1	0.4	0.7	3.9	2.10	3.4	3.8	1.10	3.3	1.3	3.7	1.8	1.4	0.10	1.9	2.7	2.4	1.1	2.8	2.6	0.2	2.5	3.2	0.6
8	0.5	3.2	2.5	0.3	0.6	3.6	0.7	0.4	3.5	3.4	2.10	1.7	3.3	1.10	3.10	1.3	3.7	0.10	1.4	1.5	2.9	2.7	0.1	2.6	2.8
9	2.9	1.1	2.8	2.6	0.2	0.5	3.2	0.6	0.3	3.1	0.4	0.7	3.9	2.10	3.4	3.8	1.7	2.2	1.3	0.8	1.8	0.9	0.10	0.0	2.7
10	0.10	0.9	1.5	2.9	2.7	0.1	1.1	2.8	0.5	0.2	2.5	0.3	0.6	3.6	0.7	0.4	1.8	3.9	2.3	1.7	1.2	1.10	0.8	1.3	2.1
11	2.2	1.3	0.8	1.8	0.9	0.10	0.0	1.5	2.9	1.1	0.1	2.6	0.2	0.5	3.2	1.4	0.3	1.9	3.6	2.4	3.9	1.6	3.4	1.2	1.7
12	3.9	2.3	1.7	1.2	1.10	0.8	1.3	2.1	1.8	0.9	1.5	0.0	2.7	0.1	3.8	2.8	3.10	0.2	3.7	3.2	1.4	3.6	1.9	0.4	1.6
13	1.9	3.6	2.4	3.9	1.6	3.4	1.2	1.7	2.2	1.10	0.8	1.8	2.1	0.7	0.0	3.5	2.9	3.8	0.1	3.3	2.8	3.10	3.2	3.7	0.3
14	0.2	3.7	3.2	1.4	3.6	1.9	0.4	1.6	3.9	2.3	3.4	1.2	2.5	2.2	0.6	2.1	3.1	0.9	3.5	0.0	2.10	2.9	3.8	2.8	3.3
15	2.10	0.1	3.3	2.8	3.10	3.2	3.7	0.3	1.9	3.6	2.4	2.7	1.6	2.6	2.3	0.5	2.2	2.5	0.8	3.1	2.1	0.7	0.9	3.5	2.9
16	0.9	0.7	0.0	2.10	2.9	3.8	2.8	3.3	0.2	3.7	1.8	1.4	0.10	1.9	2.7	2.4	1.1	2.3	2.6	1.2	2.5	2.2	0.6	0.8	3.1
17	2.5	0.8	0.6	2.1	0.7	0.9	3.5	2.9	2.10	1.7	3.3	1.10	3.10	1.3	3.7	0.10	1.4	1.5	2.4	2.7	1.6	2.6	2.3	0.5	1.2
18	1.6	2.6	1.2	0.5	2.2	0.6	0.8	3.1	0.9	0.7	3.9	2.10	3.4	3.8	1.7	3.3	1.3	3.10	1.8	1.4	0.10	1.9	2.7	2.4	1.1
19	1.5	1.9	2.7	1.6	1.1	2.3	0.5	0.2	2.5	0.3	0.6	3.6	0.7	0.4	3.5	3.9	2.10	1.7	3.8	1.10	3.10	1.3	3.7	0.10	1.4
20	3.10	1.8	3.7	0.10	1.9	1.5	2.9	1.1	0.1	2.6	0.2	0.5	3.2	0.6	0.3	3.1	3.6	0.7	3.9	3.5	3.4	3.8	1.7	3.3	1.3
21	1.7	3.8	1.10	3.3	1.3	2.1	1.8	0.9	1.5	0.0	2.7	0.1	1.1	2.8	0.5	0.2	2.5	3.2	0.6	3.6	3.1	0.4	3.5	3.9	2.10
22	0.7	3.9	3.5	3.4	2.3	1.7	2.2	1.10	0.8	1.8	2.1	0.10	0.0	1.5	2.9	1.1	0.1	2.6	2.8	0.5	3.2	2.5	0.3	3.1	3.6
23	3.2	0.6	3.6	1.9	0.4	2.4	3.9	2.3	3.4	1.2	1.10	2.2	1.3	2.1	1.8	0.9	1.5	0.0	2.7	2.9	1.1	2.8	2.6	0.2	2.5
24	2.6	2.8	3.10	3.2	3.7	0.3	1.4	3.6	2.4	0.4	1.6	3.4	2.3	1.7	2.2	1.10	0.8	1.8	2.1	0.10	0.9	1.5	2.9	2.7	0.1

$< \,|\, -w \,|\, \overline{29} \,|\, \overset{+}{1+6i}$

$r_1 \backslash r_2$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
0	0.0	0.7	2.3	3.9	1.4	1.3	0.2	0.9	1.8	2.8	0.4	3.6	1.7	2.6	1.2
1	0.9	2.1	1.2	3.7	3.6	1.7	1.6	0.1	0.8	0.6	2.9	1.1	3.3	0.3	3.8
2	0.1	0.8	2.2	1.5	3.4	3.3	0.3	1.9	0.0	0.7	0.5	3.9	1.4	1.3	0.2
3	1.4	0.0	0.7	2.3	1.8	3.1	1.3	0.2	0.9	2.1	1.2	0.4	3.6	1.7	1.6
4	2.6	1.7	2.1	1.2	3.7	0.6	2.4	1.6	0.1	0.8	2.2	1.5	1.1	3.3	0.3
5	1.1	3.8	0.3	2.2	1.5	3.4	0.5	2.5	1.9	0.0	0.7	2.3	1.8	1.4	1.3
6	2.5	1.4	3.5	0.2	2.3	1.8	3.1	0.4	2.6	0.9	2.1	1.2	3.7	0.6	1.7
7	0.4	2.6	1.7	3.2	0.1	3.7	0.6	2.4	1.1	3.8	0.8	2.2	1.5	3.4	0.5
8	3.3	1.1	3.8	0.3	2.7	0.0	3.4	0.5	2.5	1.4	3.5	0.7	2.3	1.8	3.1
9	2.3	1.3	1.4	3.5	0.2	2.8	2.1	3.1	0.4	2.6	1.7	3.2	1.2	3.7	0.6
10	3.6	3.7	1.6	1.7	3.2	0.1	2.9	2.2	2.4	1.1	3.8	0.3	2.7	1.5	3.4
11	2.2	3.3	3.4	1.9	0.3	2.7	0.0	3.9	2.3	2.5	1.4	3.5	0.2	2.8	1.3
12	3.9	2.3	1.3	3.1	0.9	0.2	2.8	2.1	3.6	3.7	2.6	1.7	3.2	0.1	2.9
13	3.2	3.6	3.7	1.6	2.4	0.8	0.1	2.9	2.2	3.3	3.4	3.8	0.3	2.7	0.0
14	1.5	2.7	3.3	3.4	1.9	2.5	0.7	0.0	3.9	2.3	1.3	3.1	3.5	0.2	2.8
15	3.5	1.8	2.8	1.3	3.1	0.9	2.6	1.2	2.1	3.6	3.7	1.6	2.4	3.2	0.1
16	1.2	3.2	0.6	2.9	1.6	2.4	0.8	3.8	1.5	2.2	3.3	3.4	1.9	2.5	1.9
17	1.9	1.5	2.7	0.5	3.9	1.9	2.5	0.7	3.5	1.8	2.3	1.3	3.1	0.4	2.6
18	0.2	1.9	1.8	2.8	0.4	3.6	0.9	2.6	1.2	3.2	0.6	3.7	0.6	2.4	1.1
19	1.6	0.1	0.8	0.6	2.9	1.1	3.3	0.8	3.8	1.5	2.7	0.0	3.4	0.5	2.5
20	0.3	1.9	0.0	0.7	0.5	3.9	1.4	1.3	0.7	3.5	0.2	2.8	2.1	3.1	0.4
21	1.3	0.2	0.9	2.1	1.2	0.4	3.6	1.7	1.6	1.7	3.2	0.1	2.9	2.2	2.4
22	2.4	1.5	0.1	0.8	2.2	1.5	1.1	3.3	3.4	1.9	0.3	2.7	0.0	3.9	2.3
23	0.5	2.5	1.9	0.0	0.7	2.3	1.8	2.3	1.3	3.1	0.9	0.2	2.8	2.1	3.6
24	3.1	0.4	2.6	0.9	2.1	1.2	3.2	0.6	3.7	1.6	2.4	0.8	0.1	2.9	2.2
25	0.6	2.4	1.1	3.8	0.8	3.8	1.5	2.7	0.5	3.4	1.9	2.5	0.7	0.0	3.9
26	3.4	0.5	2.5	1.4	1.3	0.7	3.5	1.8	2.8	0.4	3.1	0.9	2.6	1.2	2.1
27	2.1	3.1	0.4	3.6	1.7	1.6	1.2	3.2	0.6	2.9	1.1	2.4	0.8	3.8	1.5
28	2.9	2.2	2.9	1.1	3.3	0.3	1.9	1.5	2.7	0.5	3.9	1.4	2.5	0.7	3.5

< | —w | $\bar{29}$ | $\frac{+}{1-6i}$

r_1 \ r_2	15	16	17	18	19	20	21	22	23	24	25	26	27	28
0	3.2	0.6	3.7	1.6	2.4	0.8	3.8	2.9	2.2	3.3	3.4	1.0	0.3	2.7
1	1.5	2.7	0.5	3.4	1.9	2.5	0.7	3.5	3.9	2.3	1.3	3.1	0.9	0.2
2	3.5	1.8	2.8	0.4	3.1	0.9	2.6	1.2	3.2	3.6	3.7	1.6	2.4	1.1
3	0.1	3.2	0.6	2.9	1.1	2.4	0.8	3.8	1.5	2.7	3.3	3.4	0.5	2.5
4	1.9	0.0	2.7	0.5	3.9	1.4	2.5	0.7	3.5	1.8	2.8	1.8	3.1	0.4
5	0.2	0.9	2.1	2.8	0.4	3.6	1.7	2.6	1.2	3.2	0.1	2.9	0.6	2.4
6	1.6	0.1	0.8	2.2	2.9	1.1	3.3	0.3	3.8	0.3	2.7	0.0	3.9	0.5
7	0.3	1.9	0.0	0.7	2.3	3.9	1.4	1.3	3.1	3.5	0.2	2.8	2.1	3.6
8	0.4	0.2	0.9	2.1	1.2	3.7	3.6	3.7	1.6	2.4	3.2	[illegible]	2.9	2.2
9	2.4	1.1	0.1	0.8	2.2	1.5	2.7	3.3	3.4	1.9	2.5	2.7	0.0	3.9
10	0.5	2.5	1.4	0.0	0.7	3.5	1.8	2.8	1.3	3.1	0.9	2.6	2.6	2.1
11	3.1	0.4	2.6	1.7	2.6	1.2	3.2	0.6	2.9	1.6	2.4	0.8	3.8	2.9
12	0.6	2.4	1.1	3.3	0.3	3.8	1.5	2.7	0.5	3.9	1.9	2.5	0.7	3.5
13	3.9	0.5	3.9	1.4	1.3	0.2	3.5	1.8	2.8	0.4	3.6	0.9	2.6	1.2
14	2.1	1.2	0.4	3.6	1.7	1.6	0.1	3.2	0.6	2.9	1.1	3.3	0.8	3.8
15	0.8	2.2	1.5	1.1	3.3	0.3	1.9	0.0	2.7	0.5	3.9	1.4	1.3	0.7
16	0.0	0.7	2.3	1.8	1.4	1.3	0.2	0.9	2.1	2.8	0.4	3.6	1.7	1.6
17	0.9	2.1	1.2	3.7	0.6	1.7	1.6	0.1	0.8	2.2	2.9	1.1	3.3	0.3
18	3.8	0.8	2.2	1.5	3.4	0.5	0.3	1.9	0.0	0.7	2.3	3.9	1.4	1.3
19	1.4	3.4	0.7	2.3	1.8	3.1	0.4	0.2	0.9	2.1	1.2	3.7	3.6	1.7
20	2.6	1.7	3.2	1.2	3.7	0.6	2.4	1.1	0.1	0.8	2.2	1.5	3.4	3.3
21	1.1	3.8	0.3	2.7	1.5	3.4	0.5	2.5	1.4	0.0	0.7	2.3	1.8	3.1
22	2.5	1.4	3.5	0.2	2.8	1.8	3.1	0.4	2.6	1.7	2.1	1.2	3.7	0.6
23	3.7	2.6	1.7	3.2	0.1	2.9	0.6	2.4	1.1	3.8	0.3	2.2	1.5	3.4
24	3.3	3.4	3.8	0.3	2.7	0.0	3.9	0.5	2.5	1.4	3.5	0.2	2.3	1.8
25	2.3	1.3	3.1	3.5	0.2	2.8	2.1	3.6	0.4	2.6	1.7	3.2	0.1	3.7
26	3.6	3.7	1.6	2.4	3.2	0.1	2.9	2.2	3.3	1.1	3.8	0.3	2.7	0.0
27	2.2	3.3	3.4	1.9	2.5	2.7	0.0	3.9	2.3	1.3	1.4	3.5	0.2	2.8
28	1.8	2.3	1.3	3.1	0.9	2.6	2.8	2.1	3.6	3.7	1.6	1.7	3.2	0.1

< | —w | $\bar{29}$ | $\frac{+}{1-6i}$

r_1 \ r_2	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
0	0.0	3.9	0.1	2.4	1.8	3.1	1.6	3.4	1.3	2.3	0.8	2.2	0.9	1.2	2.6
1	3.5	2.1	3.7	0.0	2.5	0.7	2.4	1.8	3.1	3.3	0.5	1.3	2.3	0.8	1.5
2	2.3	3.2	2.2	3.5	2.1	0.3	0.6	2.5	0.7	2.4	2.8	0.4	3.3	0.5	1.3
3	0.9	0.5	2.6	2.3	3.2	2.2	0.2	1.2	0.3	0.6	2.5	2.9	1.1	2.8	0.4
4	2.4	0.8	0.4	2.7	0.5	2.6	2.3	0.1	1.5	0.2	1.2	0.3	3.9	1.4	2.9
5	3.3	2.5	1.3	1.1	3.8	0.4	2.7	0.5	0.0	1.7	0.1	1.5	0.2	3.7	1.6
6	0.2	2.8	0.3	3.3	1.4	3.6	1.1	3.8	0.4	2.1	1.9	0.0	1.7	0.1	3.5
7	3.9	0.1	2.9	0.2	2.8	1.6	3.4	1.4	3.6	1.1	2.2	0.9	2.1	1.9	0.0
8	2.7	3.7	0.0	3.9	0.1	2.9	1.8	3.1	1.6	3.4	1.4	2.3	0.8	2.2	0.9
9	1.7	3.8	3.5	2.1	3.7	0.0	3.9	0.7	2.4	1.8	3.1	1.6	0.5	1.3	2.3
10	3.4	1.9	3.6	3.2	2.2	3.5	2.1	3.7	0.6	2.5	0.7	2.4	1.8	0.4	3.3
11	0.8	3.1	0.9	3.4	2.6	2.3	3.2	2.2	3.5	1.2	0.3	0.6	2.5	0.7	1.1
12	2.5	1.3	2.4	0.8	3.1	2.7	0.5	2.6	2.3	3.2	1.5	0.2	1.2	0.3	0.6
13	1.8	0.3	3.3	2.5	1.3	2.4	3.8	0.4	2.7	0.5	2.6	1.7	0.1	1.5	0.2
14	3.2	0.7	0.2	2.8	0.3	3.3	2.5	3.6	1.1	3.8	0.4	2.7	1.9	0.0	1.7
15	1.2	2.6	0.6	0.1	2.9	0.2	2.8	0.3	3.4	1.4	3.6	1.1	3.8	0.9	2.1
16	3.8	1.5	2.7	1.2	0.0	3.9	0.1	2.9	0.2	3.1	1.6	3.4	1.4	3.6	3.2
17	0.5	3.6	1.7	3.8	1.5	2.1	3.7	0.0	3.9	0.1	2.4	1.8	3.1	0.9	3.4
18	2.8	0.4	3.4	1.9	3.6	1.7	2.2	3.5	2.1	3.7	0.0	2.5	1.3	2.4	0.8
19	1.4	2.9	1.1	3.1	0.9	3.4	1.9	2.3	3.2	2.2	3.5	1.8	0.3	3.3	2.5
20	3.7	1.6	3.9	1.4	2.4	0.8	3.1	0.9	0.5	2.6	0.6	3.2	0.7	0.2	2.8
21	0.7	3.5	1.8	3.7	1.6	2.5	1.3	2.4	0.8	1.5	2.7	1.2	2.6	0.6	0.1
22	1.9	0.6	3.2	0.7	3.5	1.8	0.3	3.3	0.5	1.3	1.7	3.8	1.5	2.7	1.2
23	2.2	0.9	1.2	2.6	0.6	3.2	0.7	1.1	2.8	0.4	3.3	1.9	3.6	1.7	3.8
24	1.3	2.3	0.8	1.5	2.7	1.2	0.3	0.6	1.4	2.9	1.1	2.8	0.9	3.4	1.9
25	0.4	3.3	0.5	1.3	1.7	0.1	1.5	0.2	1.2	1.6	3.9	1.4	2.9	0.8	3.1
26	2.9	1.1	2.8	0.4	2.7	1.9	0.0	1.7	0.1	1.5	1.8	3.7	1.6	3.9	1.3
27	0.3	3.9	1.4	3.6	1.1	3.8	0.9	2.1	1.9	0.0	1.7	0.7	3.5	1.8	3.7
28	1.5	0.2	3.1	1.6	3.4	1.4	3.6	0.8	2.2	0.9	2.1	1.9	0.6	3.2	0.7

$< \mid -w \mid \overline{29} \mid \overset{+}{1-6l}$

r_1 \ r_2	15	16	17	18	19	20	21	22	23	24	25	26	27	28
0	0.6	3.2	2.9	0.2	2.8	0.3	3.3	1.4	3.6	1.1	3.8	0.6	2.1	1.9
1	2.7	1.2	2.6	3.9	0.1	2.9	0.2	2.8	1.6	3.4	1.4	3.6	1.1	2.2
2	1.7	3.8	1.5	2.7	3.7	0.0	3.9	0.1	2.9	1.8	3.1	1.8	3.4	1.9
3	3.3	1.9	3.6	1.7	3.8	3.5	2.1	3.7	0.0	3.9	0.7	2.4	0.8	[illegible]
4	1.1	2.8	0.9	3.4	1.9	3.6	3.2	2.2	3.5	2.1	3.7	3.3	2.5	1.3
5	3.9	1.4	2.9	0.8	3.1	0.9	3.4	2.6	2.3	3.2	0.7	3.5	2.8	0.3
6	1.8	3.7	1.6	3.9	1.3	2.4	0.8	3.1	2.7	1.2	2.6	0.6	3.2	2.9
7	3.2	0.7	3.5	1.8	3.7	3.3	2.5	1.3	2.3	3.8	1.5	2.7	1.2	2.6
8	2.1	2.6	0.6	3.2	0.7	3.5	2.8	0.4	3.3	0.5	3.6	1.7	3.3	1.5
9	0.8	2.2	2.7	1.2	2.6	0.6	2.5	2.9	1.1	2.8	0.4	3.4	1.9	3.5
10	0.5	1.3	2.3	3.8	1.5	0.2	1.2	0.3	3.9	1.4	2.9	1.1	3.1	0.9
11	2.8	0.4	3.3	0.5	0.0	1.7	0.1	1.5	0.2	3.7	1.6	3.9	1.4	2.4
12	1.4	2.9	1.1	3.8	0.4	2.1	1.9	0.0	1.7	0.1	3.5	1.8	3.7	1.6
13	1.2	1.6	3.4	1.4	3.6	1.1	2.2	0.9	2.1	1.9	0.0	3.2	0.7	3.5
14	0.1	2.9	1.8	3.1	1.6	3.4	1.4	2.3	0.8	2.2	0.9	2.1	2.6	0.6
15	3.7	0.0	3.9	0.7	2.4	1.8	3.1	1.6	0.5	1.3	2.3	0.8	2.2	2.7
16	2.2	3.5	2.1	3.7	0.6	2.5	0.7	2.4	1.8	0.4	3.3	0.5	1.3	2.3
17	2.6	2.3	3.2	2.2	3.5	1.2	0.3	0.6	2.5	0.7	1.1	2.8	0.4	3.3
18	3.1	2.7	0.5	2.6	2.3	3.2	1.5	0.2	1.2	0.3	0.6	1.4	2.9	1.1
19	1.3	2.4	3.8	0.4	2.7	0.5	2.6	1.7	0.1	1.5	0.2	1.2	1.6	3.9
20	0.3	3.3	2.5	3.6	1.1	3.8	0.4	2.7	1.9	0.0	1.7	0.1	1.5	1.8
21	2.9	0.2	2.8	0.3	3.4	1.4	3.6	1.1	3.8	0.9	2.1	1.9	0.0	1.7
22	0.0	3.9	0.1	2.9	0.2	3.1	1.6	3.4	1.4	3.6	0.8	2.2	0.9	2.1
23	1.5	2.1	3.7	0.0	3.9	0.1	2.4	1.8	3.1	1.6	3.4	1.3	2.3	0.8
24	3.6	1.7	2.2	3.5	2.1	3.7	0.0	2.5	0.7	2.4	1.8	3.1	3.3	0.5
25	0.9	3.4	1.9	2.3	3.2	2.2	3.5	2.1	0.3	0.6	2.5	0.7	2.4	2.8
26	2.4	0.8	3.1	0.9	0.5	2.6	2.3	3.2	2.2	0.2	1.2	0.3	0.6	2.5
27	3.3	2.5	1.3	2.4	0.8	0.4	2.7	0.5	2.6	2.3	0.1	1.5	0.2	1.2
28	3.5	2.8	0.3	3.3	2.5	1.3	1.1	3.8	0.4	2.7	0.5	0.0	1.7	0.1

$< \mid -w \mid \overline{29} \mid \overset{+}{4+5l}$

r_1 \ r_2	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
0	0.0	0.1	1.7	1.8	0.2	0.3	2.9	2.8	0.4	1.4	3.2	3.6	1.1	2.6	3.9
1	3.3	2.1	0.0	1.7	1.10	0.1	0.2	0.10	2.9	0.3	0.4	2.8	3.2	1.5	2.7
2	0.9	2.10	2.2	2.1	3.10	3.7	0.0	0.1	1.3	0.10	0.2	0.3	2.9	2.8	1.8
3	2.4	0.8	0.9	2.3	2.2	3.8	3.3	2.1	0.0	1.7	1.3	0.1	0.2	0.10	2.9
4	0.7	3.4	1.2	0.8	2.4	2.3	3.5	2.10	2.2	2.1	3.10	1.7	0.0	0.1	1.3
5	1.9	0.6	0.7	1.6	1.2	3.4	2.4	3.1	0.9	2.3	2.2	3.8	3.10	2.1	0.0
6	3.6	3.9	0.5	0.6	1.9	1.6	0.7	3.4	2.5	0.8	2.4	2.3	3.5	3.8	2.2
7	1.4	3.2	3.6	1.1	0.5	3.9	1.9	0.6	0.7	2.6	1.2	3.4	2.4	3.1	3.5
8	1.10	0.4	2.8	3.2	1.5	1.1	3.6	3.9	0.5	0.6	2.7	1.6	0.7	3.4	2.5
9	0.10	3.7	0.3	2.9	2.8	1.8	1.5	3.2	3.6	1.1	0.5	1.4	1.9	0.6	0.7
10	1.7	1.3	3.3	0.2	0.10	2.9	1.10	1.8	2.8	3.2	1.5	1.1	0.4	3.9	0.5
11	2.1	3.10	1.7	2.10	0.1	1.3	0.10	3.7	1.10	2.9	2.8	1.8	1.5	0.3	3.6
12	2.3	2.2	3.8	3.10	0.9	0.0	1.7	1.3	3.3	3.7	0.10	2.9	1.10	1.8	0.2
13	3.1	2.4	2.3	3.5	3.8	0.8	2.1	3.10	1.7	2.10	3.3	1.3	0.10	3.7	1.10
14	2.6	2.5	3.4	2.4	3.1	3.5	1.2	2.2	3.8	3.10	0.9	2.10	1.7	1.3	3.3
15	0.6	2.7	2.6	0.7	3.4	2.5	3.1	1.6	2.3	3.5	3.8	0.8	0.9	3.10	1.7
16	1.1	0.5	1.4	2.7	0.6	0.7	2.6	2.5	1.9	2.4	3.1	3.5	1.2	0.8	2.1
17	0.3	1.5	1.1	0.4	1.4	0.5	0.6	2.7	2.6	3.9	3.4	2.5	3.1	3.5	1.2
18	0.1	0.2	1.8	1.5	0.3	0.4	1.1	0.5	1.4	2.7	3.6	0.7	3.4	2.5	3.1
19	3.7	0.0	0.1	1.10	1.8	0.2	0.3	1.5	1.1	0.4	1.4	2.7	0.6	0.7	2.6
20	2.6	3.3	2.1	0.0	3.7	1.10	0.1	0.2	1.8	1.5	1.1	0.4	1.4	0.5	0.6
21	3.10	0.9	2.10	2.2	2.1	3.3	3.7	0.0	0.1	3.8	1.8	1.5	0.3	0.4	1.1
22	3.4	3.8	0.8	0.9	2.3	2.2	2.10	3.3	3.7	0.0	2.9	1.10	1.8	0.2	0.3
23	1.6	0.7	3.5	1.2	0.8	2.4	2.3	1.7	2.10	3.3	2.1	0.10	3.7	1.10	0.1
24	3.9	1.9	0.6	3.1	1.6	1.2	0.8	2.4	3.10	0.9	2.10	2.2	1.3	3.3	3.7
25	2.7	3.6	3.9	0.5	2.5	3.1	1.6	1.2	3.4	3.8	0.8	0.9	2.3	1.7	2.10
26	0.4	1.4	3.2	3.6	3.9	2.6	2.5	1.9	1.6	0.7	3.5	1.2	0.8	2.4	3.10
27	2.9	0.3	0.4	1.4	3.2	3.6	2.7	2.6	3.9	1.9	0.6	3.1	1.6	1.2	3.4
28	1.3	0.10	1.5	0.3	0.4	2.8	3.2	1.4	2.7	3.6	3.9	0.5	2.5	1.9	1.6

$< \mid -w \mid \overline{29} \mid \overset{+}{4+5i}$

r_1 \ r_2	15	16	17	18	19	20	21	22	23	24	25	26	27	28
0	1.9	0.6	3.1	1.6	1.2	3.4	2.4	0.8	0.9	2.3	2.2	3.8	3.3	2.1
1	3.6	3.9	0.5	2.5	1.9	1.6	0.7	3.4	1.2	0.8	2.4	2.3	3.5	2.10
2	1.4	3.2	3.6	1.1	2.6	3.9	1.9	0.6	0.7	1.6	1.2	3.4	2.4	2.3
3	1.10	0.4	2.8	3.2	1.5	2.7	3.6	3.9	0.5	0.6	1.9	1.6	1.2	3.4
4	0.10	3.7	0.3	2.9	2.8	1.8	1.4	3.2	3.6	1.1	0.5	2.5	1.9	1.6
5	1.7	1.3	3.3	0.2	0.10	2.9	1.10	0.4	2.8	3.2	3.6	1.1	2.6	3.9
6	2.1	3.10	1.7	2.10	0.1	1.3	0.10	3.7	0.3	0.4	2.8	3.2	1.5	2.7
7	2.3	2.2	3.8	3.10	0.9	0.0	1.7	1.3	0.10	0.2	0.3	2.9	2.8	1.8
8	3.1	2.4	2.3	3.5	3.8	0.8	2.1	0.0	1.7	1.3	0.1	0.2	0.10	2.9
9	2.6	2.5	3.4	2.4	3.1	3.5	3.8	2.2	2.1	3.10	1.7	0.0	0.1	1.3
10	0.6	2.7	2.6	0.7	3.4	2.4	3.1	3.5	2.3	2.2	3.8	3.10	2.1	0.0
11	1.1	0.5	1.4	2.7	1.6	0.7	3.4	2.5	3.1	2.4	2.3	3.5	3.8	2.2
12	3.2	1.5	1.1	0.5	1.4	1.9	0.6	0.7	2.6	2.5	3.4	2.4	3.1	3.5
13	0.1	2.8	3.2	1.5	1.1	0.4	3.9	0.5	0.6	2.7	2.6	0.7	3.4	2.5
14	3.7	1.10	2.9	2.8	1.8	1.5	0.3	3.6	1.1	0.5	1.4	2.7	0.6	0.7
15	1.3	3.3	3.7	0.10	2.9	1.10	1.8	0.2	3.2	1.5	1.1	0.4	1.4	0.5
16	3.10	1.7	2.10	3.3	1.3	0.10	3.7	1.10	0.1	2.8	1.8	1.5	0.3	0.4
17	2.2	3.8	3.10	0.9	2.10	1.7	1.3	3.3	3.7	0.0	2.9	1.10	1.8	0.2
18	1.6	2.3	3.5	3.8	0.8	0.9	3.10	1.7	2.10	3.3	2.1	0.10	3.7	1.10
19	2.5	1.9	2.4	3.1	3.5	1.2	0.8	3.8	3.10	0.9	2.10	2.2	1.8	3.3
20	2.7	2.6	3.9	3.4	2.5	3.1	1.6	1.2	3.5	3.8	0.8	0.9	2.3	1.7
21	0.5	1.4	2.7	3.6	0.7	2.6	2.5	1.9	1.6	3.1	3.5	1.2	0.8	2.4
22	1.5	1.1	0.4	1.4	3.2	0.6	2.7	2.6	3.9	1.9	2.5	3.1	1.6	1.2
23	0.2	1.8	1.5	0.3	0.4	2.8	0.5	1.4	2.7	3.6	3.9	2.6	2.5	1.9
24	0.0	0.1	1.10	1.8	0.2	0.3	2.9	1.1	0.4	1.4	3.2	3.6	2.7	2.6
25	3.3	2.1	0.0	3.7	1.10	0.1	0.2	0.10	1.5	0.3	0.4	2.8	3.2	1.4
26	0.9	2.10	2.2	2.1	3.3	3.7	0.0	0.1	1.3	1.8	0.2	0.3	2.9	2.8
27	3.8	0.8	0.9	2.3	2.2	2.10	3.3	2.1	0.0	1.7	1.10	0.1	0.2	0.10
28	0.7	3.5	1.2	0.8	2.4	2.8	0.9	2.10	2.2	2.1	3.10	3.7	0.0	0.1

$< \mid -w \mid \overline{29} \mid \overset{-}{4-5i}$

r_1 \ r_2	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
0	0.0	1.3	3.9	2.4	3.10	2.9	0.8	3.4	2.7	3.7	2.10	2.1	2.3	1.1	1.8
1	0.1	2.1	1.7	3.6	1.4	3.8	0.9	1.2	0.4	0.10	3.3	0.7	2.2	2.4	1.5
2	3.3	0.0	2.2	3.9	3.2	3.10	3.5	0.8	1.6	0.3	1.3	2.10	0.6	2.3	1.4
3	2.6	2.10	2.1	2.3	3.6	2.8	3.8	3.1	1.2	1.9	0.2	1.7	0.7	0.5	2.4
4	0.2	2.7	0.7	2.2	2.4	3.2	2.9	3.5	2.5	1.6	3.7	0.1	3.9	0.6	1.1
5	0.3	0.1	0.10	0.6	2.3	1.4	2.8	0.9	3.1	2.6	1.9	3.3	0.0	3.6	0.5
6	1.9	0.2	0.0	1.3	0.5	2.4	3.10	2.9	0.8	2.5	2.7	3.7	2.10	2.1	3.2
7	1.6	3.7	0.1	2.1	1.7	1.1	1.4	3.8	0.9	1.2	2.6	0.10	3.3	0.7	2.2
8	0.4	1.9	3.3	0.0	2.2	3.9	1.8	3.10	3.5	0.8	1.6	2.7	1.3	2.10	0.6
9	3.4	0.3	3.7	2.10	2.1	2.3	3.6	1.8	3.8	3.1	1.2	1.9	0.10	1.7	0.7
10	1.2	0.4	0.2	3.3	0.7	2.2	2.4	3.2	1.10	3.5	2.5	1.6	3.7	1.3	3.9
11	0.8	1.6	0.3	0.1	2.10	0.6	2.3	1.4	2.8	3.4	3.1	2.6	1.9	3.3	1.7
12	3.1	1.2	1.9	0.2	0.0	0.7	0.5	2.4	3.10	2.9	0.4	2.5	2.7	3.7	2.10
13	1.8	2.5	1.6	3.7	0.1	2.1	0.6	1.1	1.4	3.8	0.9	0.3	2.6	0.10	3.3
14	0.9	1.10	2.6	1.9	3.3	0.0	2.2	0.5	1.5	3.10	3.5	0.8	0.2	2.7	1.3
15	2.9	0.8	3.4	2.7	3.7	2.10	2.1	2.3	1.1	1.8	3.8	3.1	1.2	0.1	0.10
16	3.8	0.9	1.2	0.4	0.10	3.3	0.7	2.2	2.4	1.5	1.10	3.5	2.5	1.6	2.7
17	1.1	3.5	0.8	1.6	0.3	1.3	2.10	0.6	2.3	1.4	1.8	3.4	3.1	1.2	1.9
18	2.8	1.5	3.1	1.2	1.9	0.2	1.7	0.7	0.5	2.4	3.10	1.10	3.5	2.5	1.6
19	3.2	2.9	1.8	2.5	1.6	3.7	0.1	3.9	0.6	1.1	1.4	2.8	3.4	3.1	2.6
20	1.4	2.8	0.9	1.10	2.6	1.9	3.3	0.0	3.6	0.5	2.4	3.10	2.9	0.4	2.5
21	2.4	3.10	2.9	0.8	3.4	2.7	3.7	2.10	2.1	0.6	1.1	1.4	3.8	0.9	0.3
22	3.6	1.4	3.8	0.9	1.2	0.4	0.10	3.3	0.0	2.2	0.5	1.5	3.10	3.5	0.8
23	3.9	3.2	3.10	3.5	0.8	1.6	0.3	3.7	2.10	2.1	2.3	1.1	1.8	3.8	3.1
24	2.3	3.6	2.8	3.8	3.1	1.2	0.4	0.2	3.3	0.7	2.2	2.4	1.5	1.10	3.5
25	2.2	2.4	3.2	2.9	3.5	0.8	1.6	0.3	0.1	2.10	0.6	2.3	1.4	1.8	3.4
26	0.6	2.3	1.4	2.8	1.5	3.1	1.2	1.9	0.2	0.0	0.7	0.5	2.4	3.10	1.10
27	1.3	0.5	2.4	3.2	2.9	1.8	2.5	1.6	3.7	0.1	2.1	0.6	1.1	1.4	3.8
28	2.1	1.7	2.3	1.4	2.8	0.9	1.10	2.6	1.9	3.3	0.0	2.2	0.5	1.5	3.10

$< \mid -w \mid \overline{29} \mid \overset{+}{4-5i}$

r_1 \ r_2	15	16	17	18	19	20	21	22	23	24	25	26	27	28
0	3.8	3.1	0.3	0.1	0.10	1.7	0.7	1.4	2.8	0.9	1.10	0.4	1.9	3.3
1	1.10	3.5	2.5	0.2	0.0	1.3	3.9	0.6	3.10	2.9	0.8	3.4	0.3	3.7
2	1.8	3.4	3.1	2.6	0.1	2.1	1.7	3.6	0.5	3.8	0.9	1.2	0.4	2.5
3	3.10	1.10	0.4	2.5	2.7	0.0	2.2	3.9	3.2	1.1	3.5	0.8	3.4	0.3
4	1.4	3.8	3.4	0.3	2.6	0.10	2.1	2.3	3.6	2.8	1.5	0.9	1.2	0.4
5	1.5	3.10	3.5	0.4	0.2	2.7	1.3	2.2	2.4	3.2	1.1	1.8	0.8	1.6
6	1.1	1.8	3.8	3.1	0.3	0.1	0.10	1.7	2.3	3.6	2.8	1.5	1.10	1.2
7	2.8	1.5	1.10	3.5	2.5	0.2	0.0	1.3	2.10	2.4	3.2	2.9	1.8	3.4
8	2.3	2.9	1.8	3.4	3.1	2.6	0.1	0.10	1.7	0.7	1.4	2.8	0.9	1.10
9	0.5	2.4	0.9	1.10	0.4	2.5	1.6	0.0	1.3	3.9	0.6	3.10	2.9	0.8
10	0.6	1.1	1.4	0.8	3.4	3.1	2.6	1.9	2.1	1.7	3.6	0.5	3.8	0.9
11	3.6	0.5	1.5	3.10	1.10	0.4	2.5	2.7	3.7	2.2	3.9	3.2	1.1	3.5
12	3.9	3.2	1.1	1.4	8.8	3.4	0.3	2.6	0.10	3.3	2.3	3.6	2.8	1.5
13	0.7	3.6	0.5	1.5	3.10	3.5	0.4	0.2	2.7	1.3	2.10	2.4	3.2	3.9
14	2.10	2.1	3.2	1.1	1.8	3.8	3.1	0.3	0.1	0.10	1.7	0.7	1.4	2.8
15	3.3	0.7	2.2	2.8	1.5	1.10	3.5	2.5	0.2	0.0	1.3	3.9	0.6	3.10
16	1.3	2.10	0.6	2.3	2.9	1.8	3.4	3.1	2.6	0.1	2.1	1.7	3.6	0.5
17	0.10	1.7	0.7	0.5	2.4	0.9	1.10	0.4	2.5	2.7	0.0	2.2	3.9	3.2
18	3.7	1.3	3.9	0.6	1.1	1.4	0.8	3.4	0.3	2.6	0.10	2.1	2.3	3.6
19	1.9	3.3	1.7	3.6	0.5	1.5	3.10	1.2	0.4	0.2	2.7	1.3	2.2	2.4
20	2.7	3.7	2.10	3.9	3.2	1.1	1.8	3.8	1.6	0.3	0.1	0.10	1.7	2.3
21	2.6	0.10	3.3	0.7	3.6	2.8	1.5	1.10	3.5	1.9	0.2	0.0	1.3	3.9
22	0.2	2.7	1.3	2.10	0.6	3.2	2.9	1.8	3.4	3.1	8.7	0.1	2.1	1.7
23	1.2	0.1	0.10	1.7	0.7	0.5	2.8	0.9	1.10	0.4	2.5	3.3	0.0	2.2
24	2.5	1.6	0.0	1.3	3.9	0.6	1.1	2.9	0.8	3.4	0.3	2.6	2.10	2.1
25	3.1	2.6	1.9	2.1	1.7	3.6	0.5	1.5	0.9	1.2	0.4	0.2	2.7	0.7
26	0.4	2.5	2.7	3.7	2.2	3.9	3.2	1.1	1.8	0.8	1.6	0.3	0.1	0.10
27	3.4	0.3	2.6	0.10	8.3	2.3	3.6	2.8	1.5	1.10	1.2	1.9	0.2	0.0
28	3.5	0.4	0.2	2.7	1.3	2.10	2.4	3.2	2.9	1.8	3.4	1.6	3.7	0.1

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$< \mid -w \mid \overline{37} \mid \overset{+}{4+5i}$

r_1 \ r_3	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
0	0.0	0.4	3.10	2.9	0.8	3.7	3.4	0.7	0.2	2.7	1.3	3.2	2.10	1.8	2.3	0.1	3.1
1	1.1	2.1	0.3	3.8	0.10	1.2	3.3	0.7	0.6	0.1	1.4	1.7	2.8	0.9	1.10	2.4	0.0
2	3.3	1.5	2.2	0.2	3.5	1.3	1.6	2.10	0.6	0.5	0.0	0.4	3.10	2.9	0.8	3.7	3.4
3	3.2	2.10	1.8	2.3	0.1	3.1	1.7	1.9	0.9	0.5	1.1	2.1	0.3	3.8	0.10	1.2	3.3
4	1.7	2.8	0.9	1.10	2.4	0.0	2.5	3.10	3.9	0.8	1.1	1.5	2.2	0.2	3.5	1.3	1.6
5	0.4	3.10	2.9	0.8	3.7	3.4	2.1	2.6	3.8	3.6	1.2	1.5	1.8	2.3	0.1	3.1	1.7
6	2.1	0.3	3.8	0.10	1.2	3.3	0.7	2.2	2.7	3.5	3.2	1.6	1.8	1.10	2.4	0.0	2.5
7	3.4	2.2	0.2	3.5	1.3	1.6	2.10	0.6	2.3	1.4	3.1	2.8	1.9	1.10	3.7	3.4	2.1
8	1.5	0.7	2.3	0.1	3.1	1.7	1.9	0.9	0.5	2.4	0.4	2.5	2.9	3.9	3.7	3.3	0.7
9	1.6	1.8	0.6	2.4	0.0	2.5	3.10	3.9	0.8	1.1	3.4	0.3	2.6	0.10	3.6	3.3	2.10
10	2.8	1.9	1.10	0.5	3.4	2.1	2.6	3.8	3.6	1.2	1.5	0.7	0.2	2.7	1.3	3.2	2.10
11	2.5	2.9	3.9	3.7	1.1	0.7	2.2	2.7	3.5	3.2	1.6	1.8	0.6	0.1	1.4	1.7	2.8
12	0.3	2.6	0.15	3.6	3.3	1.5	0.6	2.3	1.4	3.1	2.8	1.9	1.10	0.5	0.0	0.4	3.10
13	0.7	0.2	2.7	1.3	3.2	2.10	1.8	0.5	2.4	0.4	2.5	2.9	3.9	3.7	1.1	2.1	0.3
14	1.8	0.6	0.1	1.4	1.7	2.8	0.9	1.10	1.1	3.4	0.3	2.6	0.10	3.6	3.3	1.5	2.2
15	1.9	1.10	0.5	0.0	0.4	3.10	2.9	0.8	3.7	1.5	0.7	0.2	2.7	1.3	3.2	2.10	1.8
16	2.9	3.9	3.7	1.1	2.1	0.3	3.8	0.10	1.2	3.3	1.8	0.6	0.1	1.4	1.7	2.8	0.9
17	2.6	0.10	3.6	3.3	1.5	2.2	0.2	3.5	1.3	1.6	2.10	1.10	0.5	0.0	0.4	3.10	2.9
18	0.2	2.7	1.3	3.2	2.10	1.8	2.3	0.1	3.1	1.7	1.9	0.9	3.7	1.1	2.1	0.3	3.8
19	2.2	0.1	1.4	1.7	2.8	0.9	1.10	2.4	0.0	2.5	3.10	3.9	0.8	3.3	1.5	2.2	0.2
20	0.6	2.3	0.0	0.4	3.10	2.9	0.8	3.7	3.4	2.1	2.6	3.8	3.6	1.2	2.10	1.8	2.3
21	0.9	0.5	2.4	2.1	0.3	3.8	0.10	1.2	3.3	0.7	2.2	2.7	3.5	3.2	1.6	0.9	1.10
22	3.9	0.8	1.1	3.4	2.2	0.2	3.5	1.3	1.6	2.10	0.6	2.3	1.4	3.1	2.8	1.9	3.9
23	3.8	3.6	1.2	1.5	0.7	2.3	0.1	3.1	1.7	1.9	0.9	0.5	2.4	0.4	2.5	3.10	3.9
24	2.7	3.5	3.2	1.6	1.8	0.6	2.4	0.0	2.5	3.10	3.9	0.8	1.1	3.4	2.1	2.6	3.8
25	2.3	1.4	3.1	2.8	1.9	1.10	0.5	3.4	2.1	2.6	3.8	3.6	1.2	3.3	0.7	2.2	2.7
26	0.5	2.4	0.4	2.5	2.9	3.9	3.7	1.1	0.7	2.2	2.7	3.5	1.3	1.6	2.10	0.6	2.3
27	0.8	1.1	3.4	0.3	2.6	0.10	3.6	3.3	1.5	0.6	2.3	0.1	8.1	1.7	1.9	0.9	0.5
28	3.6	1.2	1.5	0.7	0.2	2.7	1.3	3.2	2.10	1.8	1.10	2.4	0.0	2.5	8.10	3.9	0.8

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Continuation of table $< | -w | \overline{37} | {}^{+}_{4+5l}$

29	3.5	3.2	1.6	1.8	0.6	0.1	1.4	1.7	2.8	1.9	1.10	3.7	3.4	2.1	2.6	3.3	3.6
30	1.4	3.1	2.8	1.9	1.10	0.5	0.0	0.4	2.5	2.9	3.9	3.7	3.3	0.7	2.2	2.7	3.5
31	0.1	0.4	2.5	2.9	3.9	3.7	1.1	3.4	0.3	2.6	0.10	3.6	3.3	2.10	0.6	2.3	1.4
32	2.4	0.0	0.3	2.6	0.10	3.6	1.2	1.5	0.7	0.2	2.7	1.3	3.2	2.10	0.9	0.5	2.4
33	3.7	3.4	2.1	0.2	2.7	3.5	3.2	1.6	1.8	0.6	0.1	1.4	1.7	2.8	0.9	0.8	1.1
34	1.2	3.3	0.7	2.2	2.3	1.4	3.1	2.8	1.9	1.10	0.5	0.0	0.4	3.10	2.9	0.8	1.2
35	1.3	1.6	2.10	1.8	2.3	2.4	0.4	2.5	2.9	3.9	3.7	1.1	2.1	0.3	3.8	0.10	1.2
36	3.1	1.7	2.8	0.9	1.10	2.4	3.4	0.3	2.6	0.10	3.6	3.3	1.5	2.2	0.2	3.5	1.3

$< | -w | \overline{37} | {}^{+}_{4+5l}$

r_1 \ r_2	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36
0	1.3	1.9	3.9	3.7	1.1	2.1	0.3	3.8	0.10	1.2	3.3	0.7	2.2	2.7	1.4	1.7	2.8	0.9	1.10	2.4
1	2.5	1.7	3.9	3.6	3.3	1.5	2.2	0.2	3.5	1.3	1.6	2.10	0.6	2.3	1.4	0.4	3.10	2.9	0.8	3.7
2	2.1	2.6	3.8	3.6	3.2	2.10	1.8	2.3	0.1	3.1	1.7	1.9	0.9	0.5	2.4	0.4	0.3	3.8	0.10	3.6
3	0.7	2.2	2.7	3.5	3.2	2.8	0.9	1.10	2.4	0.0	2.5	3.10	3.9	0.8	1.1	3.4	0.3	0.2	2.7	1.3
4	2.10	0.6	2.3	1.4	3.1	2.8	2.9	0.8	3.7	3.4	2.1	2.6	3.8	3.6	1.2	1.5	0.7	2.2	0.1	1.4
5	1.9	0.9	0.5	2.4	0.4	2.5	2.9	0.10	1.2	3.3	0.7	2.2	2.7	3.5	3.2	1.6	2.10	0.6	2.3	0.0
6	3.10	3.9	0.8	1.1	3.4	0.3	2.6	0.10	1.3	1.6	2.10	0.6	2.3	1.4	3.1	1.7	1.9	0.9	0.5	2.4
7	2.6	3.8	3.6	1.2	1.5	0.7	0.2	2.7	1.3	1.7	1.9	0.9	0.5	2.4	0.0	2.5	3.10	3.9	0.8	1.1
8	2.2	2.7	3.5	3.2	1.6	1.8	0.6	0.1	1.4	1.7	3.10	3.9	0.8	3.7	3.4	2.1	2.6	3.8	3.6	1.2
9	0.6	2.3	1.4	3.1	2.8	1.9	1.10	0.5	0.0	0.4	3.10	3.8	0.10	1.2	3.3	0.7	2.2	2.7	3.5	3.2
10	0.9	0.5	2.4	0.4	2.5	2.9	3.9	3.7	1.1	2.1	0.3	2.6	3.5	1.3	1.6	2.10	0.6	2.3	1.4	3.1
11	0.9	0.8	1.1	3.4	0.3	2.6	0.10	3.6	3.3	1.5	0.7	0.2	2.7	3.1	1.7	1.9	0.9	0.5	2.4	0.4
12	2.9	0.8	1.2	1.5	0.7	0.2	2.7	1.3	3.2	1.6	1.8	0.6	0.1	1.4	2.5	3.10	3.9	0.8	1.1	3.4
13	3.8	0.10	1.2	1.6	1.8	0.6	0.1	1.4	3.1	2.8	1.9	1.10	0.5	0.0	0.4	2.6	3.8	3.6	1.2	1.5
14	0.2	3.5	1.3	1.6	1.9	1.10	0.5	2.4	0.4	2.5	2.9	3.9	3.7	1.1	2.1	0.3	2.7	3.5	3.2	1.6
15	2.3	0.1	3.1	1.7	1.9	3.9	0.8	1.1	3.4	0.3	2.6	0.10	3.6	3.3	1.5	2.2	0.2	1.4	3.1	2.8
16	1.10	2.4	0.0	2.5	3.10	2.9	3.6	1.2	1.5	0.7	0.2	2.7	1.3	3.2	2.10	1.8	2.3	0.1	0.4	2.5
17	0.8	3.7	3.4	2.1	0.3	3.8	0.10	3.2	1.6	1.8	0.6	0.1	1.4	1.7	2.8	0.9	1.10	2.4	0.0	0.3
18	0.10	1.2	3.3	1.5	2.2	0.2	3.5	1.3	2.8	1.9	1.10	0.5	0.0	0.4	3.10	2.9	0.8	3.7	3.4	2.1
19	3.5	1.3	3.2	2.10	1.8	2.3	0.1	3.1	1.7	2.9	3.9	3.7	1.1	2.1	0.3	3.8	0.10	1.2	3.3	0.7
20	0.1	1.4	1.7	2.8	0.9	1.10	2.4	0.0	2.5	3.10	0.10	3.6	3.3	1.5	2.2	0.2	3.5	1.3	1.6	2.10

Continuation of table $< | -w | \overline{37} | {}^{+}_{4-5l}$

21	0.5	0.0	0.4	3.10	2.9	0.8	3.7	3.4	2.1	2.6	3.8	1.3	3.2	2.10	1.8	2.3	0.1	3.1	1.7	1.9
22	3.7	1.1	2.1	0.3	3.8	0.10	1.2	3.3	0.7	2.2	2.7	3.5	1.7	2.8	0.9	1.10	2.4	0.0	2.5	3.10
23	3.6	3.3	1.5	2.2	0.2	3.5	1.3	1.6	2.10	0.6	2.3	1.4	3.1	3.10	2.9	0.8	3.7	3.4	2.1	2.6
24	3.6	3.2	2.10	1.8	2.3	0.1	3.1	1.7	1.9	0.9	0.5	2.4	0.4	2.5	3.8	0.10	1.2	3.3	0.7	2.2
25	3.5	3.2	2.8	0.9	1.10	2.4	0.0	2.5	3.10	3.9	0.8	1.1	3.4	0.3	2.6	3.5	1.3	1.6	2.10	0.6
26	1.4	3.1	2.8	2.9	0.8	3.7	3.1	2.1	2.6	3.8	3.6	1.2	1.5	0.7	0.2	2.7	3.1	1.7	1.9	0.9
27	2.4	0.4	2.5	2.9	0.10	1.2	3.3	2.5	2.2	2.7	3.5	3.2	1.6	1.8	0.6	0.1	1.4	2.5	3.10	3.9
28	1.1	3.4	0.3	2.6	0.10	1.3	1.6	2.10	2.6	2.3	1.4	3.1	2.8	1.9	1.10	0.5	0.0	0.4	2.6	3.8
29	1.2	1.5	0.7	0.2	2.7	1.3	1.7	1.9	0.9	2.7	2.4	0.4	2.5	2.9	3.9	3.7	1.1	2.1	0.3	2.7
30	3.2	1.6	1.8	0.6	0.1	1.4	1.7	3.10	3.9	0.8	1.4	3.4	0.3	2.6	0.10	3.6	3.3	1.5	2.2	0.2
31	3.1	2.8	1.9	1.10	0.5	0.0	0.4	3.10	3.8	3.6	1.2	0.4	0.7	0.2	2.7	1.3	3.2	2.10	1.8	2.3
32	0.4	2.5	2.9	3.9	3.7	1.1	2.1	0.3	3.8	3.5	3.2	1.6	1.8	0.6	0.1	1.4	1.7	2.8	0.9	1.10
33	3.4	0.3	2.6	0.10	3.6	3.3	1.5	2.2	0.2	3.5	3.1	2.8	1.9	1.10	0.5	0.0	0.4	3.10	2.9	0.8
34	1.5	0.7	0.2	2.7	1.3	3.2	2.10	1.8	2.3	0.1	3.1	2.5	2.9	3.9	3.7	1.1	2.1	0.3	3.8	0.10
35	1.6	1.8	0.6	0.1	1.4	1.7	2.8	0.9	1.10	2.4	0.0	2.5	2.6	0.10	3.6	3.3	1.5	2.2	0.2	3.5
36	1.6	1.9	1.10	0.5	0.0	0.4	3.10	2.9	0.8	3.7	3.4	2.1	2.6	2.7	1.3	3.2	2.10	1.8	2.3	0.1

$< | -w | \overline{37} | {}^{+}_{4-5l}$

r_1 \ r_2	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
0	0.0	3.1	1.3	1.2	2.10	0.4	2.1	1.4	2.5	2.8	1.6	1.5	0.3	3.10	2.6	2.9	1.9	1.8
1	0.4	2.1	2.5	1.7	1.6	0.7	0.3	2.2	3.10	2.6	2.9	1.9	1.8	0.2	3.8	2.7	0.9	3.7
2	0.7	0.3	2.2	2.6	3.9	1.9	0.6	0.2	2.3	3.8	2.7	0.9	3.7	1.10	0.1	3.5	0.10	0.8
3	1.9	0.6	0.2	2.3	2.7	3.6	3.7	0.5	0.1	2.4	3.5	0.10	0.8	3.3	3.4	0.0	3.1	1.3
4	3.6	3.7	0.5	0.1	2.4	0.10	3.2	3.3	1.1	0.0	1.4	3.1	1.3	1.2	2.10	0.4	2.1	2.5
5	0.10	3.2	3.3	1.1	0.0	1.4	1.3	2.8	2.10	1.5	2.1	3.10	2.5	1.7	1.6	0.7	0.3	2.2
6	1.4	1.3	2.8	2.10	1.5	2.1	3.10	1.7	2.9	0.7	1.8	2.2	3.8	2.6	3.9	1.9	0.6	0.2
7	3.10	3.10	1.7	2.9	0.7	1.8	2.2	3.8	3.9	0.9	0.6	1.10	2.3	3.5	2.7	3.6	3.7	0.5
8	0.2	3.8	3.8	3.9	0.9	0.6	1.10	2.3	3.5	3.6	0.8	0.5	3.4	2.4	3.1	0.10	3.2	3.3
9	1.10	0.1	3.5	3.5	3.6	0.8	0.5	3.4	2.4	3.1	3.2	1.2	1.1	0.4	1.4	2.5	1.3	2.8
10	3.3	3.4	0.0	3.1	3.1	3.2	1.2	1.1	0.4	1.4	2.5	2.8	1.6	1.5	0.8	3.10	2.6	1.7

Continuation of table $< | -w | \overline{37} | ^{+}_{4-5l}$

11	1.2	2.10	0.4	2.1	2.5	2.5	2.8	1.6	1.5	0.3	3.10	2.6	2.9	1.9	1.8	0.2	3.8	2.7
12	1.7	1.6	0.7	0.3	2.2	2.6	2.6	2.9	1.9	1.8	0.2	3.8	2.7	0.9	3.7	1.10	0.1	3.5
13	2.6	3.9	1.9	0.6	0.2	2.3	2.7	2.7	0.9	3.7	1.10	0.1	3.5	0.10	0.8	3.3	3.4	0.0
14	2.3	2.7	3.6	3.7	0.5	0.1	2.4	0.10	0.10	0.3	3.3	3.4	0.0	3.1	1.3	1.2	2.10	0.4
15	0.1	2.4	0.10	3.2	3.3	1.1	0.0	1.4	1.3	1.3	1.2	2.10	0.4	2.1	2.5	1.7	1.6	0.7
16	1.1	0.0	1.4	1.3	2.8	2.10	1.5	2.1	3.10	1.7	1.7	1.6	0.7	0.3	2.2	2.6	3.9	1.9
17	2.10	1.5	2.1	3.10	1.7	2.9	0.7	1.8	2.2	3.8	3.9	3.9	1.9	0.6	0.2	2.3	2.7	3.6
18	2.9	0.7	1.8	2.2	3.8	3.9	0.9	0.6	1.10	2.3	3.5	3.6	3.6	3.7	0.5	0.1	2.4	0.10
19	0.9	0.9	0.6	1.10	2.3	3.5	3.6	0.8	0.5	3.4	2.4	3.1	3.2	3.2	3.3	1.1	0.0	1.4
20	0.10	0.8	0.8	0.5	3.4	2.4	3.1	3.2	1.2	1.1	0.4	1.4	2.5	2.8	2.8	2.10	1.5	2.1
21	3.1	1.3	1.2	1.2	1.5	0.4	1.4	2.5	2.8	1.6	1.5	0.3	3.10	2.6	2.9	2.9	0.7	0.3
22	2.1	2.5	1.7	1.6	1.6	1.5	0.3	3.10	2.6	2.9	1.9	1.8	0.2	3.8	2.7	0.9	1.9	0.6
23	0.3	2.2	2.6	3.9	1.9	1.9	1.8	0.2	3.8	2.7	0.9	3.7	1.10	0.1	3.5	3.6	0.8	3.7
24	0.6	0.2	2.3	2.7	3.6	3.7	3.7	1.10	0.1	3.5	0.10	0.8	3.3	3.4	2.4	3.1	3.2	1.2
25	3.7	0.5	0.1	2.4	0.10	3.2	3.3	3.3	3.4	0.0	3.1	1.3	1.2	1.1	0.4	1.4	2.5	2.8
26	3.2	3.3	1.1	0.0	1.4	1.3	2.8	2.10	2.10	0.4	2.1	2.5	2.8	1.6	1.5	0.3	3.10	2.6
27	1.3	2.8	2.10	1.5	2.1	3.10	1.7	2.9	0.7	0.7	0.3	3.10	2.6	2.9	1.9	1.8	0.2	3.8
28	3.10	1.7	2.9	0.7	1.8	2.2	3.8	3.9	0.9	0.6	1.8	0.2	3.8	2.7	0.9	3.7	1.10	0.1
29	2.2	3.8	3.9	0.9	0.6	1.10	2.3	3.5	3.6	3.7	0.5	1.10	0.1	3.5	0.10	0.8	3.3	3.4
30	1.10	2.3	3.5	3.6	0.8	0.5	3.4	2.4	0.10	3.2	3.3	1.1	3.4	0.0	3.1	1.3	1.2	2.10
31	3.4	3.4	2.4	3.1	3.2	1.2	1.1	0.0	1.4	1.3	2.8	2.10	1.5	0.4	2.1	2.5	1.7	1.6
32	2.10	0.4	0.4	1.4	2.5	2.8	2.10	1.5	2.1	3.10	1.7	2.9	0.7	1.8	0.3	2.2	2.6	3.9
33	1.6	0.7	0.3	0.3	3.10	1.7	2.9	0.8	1.8	2.2	3.8	3.9	0.9	0.6	1.10	0.2	2.3	2.7
34	3.9	1.9	0.6	0.2	2.2	3.8	3.9	0.9	0.6	1.10	2.3	3.5	3.6	0.8	0.5	3.4	0.1	2.4
35	2.7	3.6	3.7	1.10	0.1	2.3	3.5	3.6	0.8	0.5	3.4	2.4	3.1	3.2	1.2	1.1	0.4	0.0
36	2.4	0.10	0.8	3.3	3.4	0.0	2.4	3.1	3.2	1.2	1.1	0.4	1.4	2.5	2.8	1.6	1.5	0.3

$< | -w | \overline{37} | ^{+}_{4-5l}$

r_1 \ r_2	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36
0	0.2	2.2	3.8	3.9	0.9	0.6	1.10	2.3	3.5	3.6	0.8	0.5	3.4	0.1	2.4	0.10	3.2	3.3	1.1
1	1.10	0.1	2.3	3.5	3.6	0.8	0.5	3.4	2.4	3.4	3.2	1.2	1.1	0.4	0.0	1.4	1.3	2.8	2.10
2	3.3	3.4	0.0	2.4	3.1	3.2	1.2	1.1	0.4	1.4	2.5	2.8	1.6	1.5	0.3	2.1	3.10	1.7	1.6
3	1.2	2.10	0.4	2.1	1.4	2.5	2.8	1.6	1.5	0.3	3.10	2.6	2.9	1.9	1.8	0.2	2.2	2.6	3.9
4	1.7	1.6	0.7	0.3	2.2	3.10	2.6	2.9	1.9	1.8	0.2	3.8	2.7	0.9	3.7	1.10	2.3	2.3	2.7

Continuation of table $< | -w | \overline{37} | ^{+}_{4-5l}$

5	2.6	3.9	1.9	0.6	0.2	2.3	3.8	2.7	0.9	3.7	1.10	0.1	3.5	0.10	0.8	0.5	3.4	2.4	2.4
6	2.3	2.7	3.6	3.7	0.5	0.1	2.4	3.5	0.10	0.8	3.3	3.4	0.0	3.1	3.2	1.2	1.1	0.4	1.4
7	0.1	2.4	0.10	3.2	3.3	1.1	0.0	1.4	3.1	1.3	1.2	2.10	0.4	1.4	2.5	2.8	1.6	1.5	0.3
8	1.1	0.0	1.4	1.3	2.8	2.10	1.5	2.1	3.10	2.5	1.7	1.6	1.5	0.3	3.10	2.6	2.9	1.9	1.8
9	2.10	1.5	2.1	3.10	1.7	2.9	0.7	1.8	2.2	3.8	2.6	2.9	1.9	1.8	0.2	3.8	2.7	0.9	3.7
10	2.9	0.7	1.8	2.2	3.8	3.9	0.9	0.6	1.10	2.3	2.7	2.7	0.9	3.7	1.10	0.1	3.5	0.10	0.8
11	3.9	0.9	0.6	1.10	2.3	3.5	3.6	0.8	0.5	0.1	2.4	0.10	0.10	0.8	3.3	3.4	0.0	3.1	1.3
12	0.10	3.6	0.8	0.5	3.4	2.4	3.1	3.2	3.3	1.1	0.0	1.4	1.3	1.3	1.2	2.10	0.4	2.1	2.5
13	3.1	1.3	3.2	1.2	1.1	0.4	1.4	1.3	2.8	2.10	1.5	2.1	3.10	1.7	1.7	1.6	0.7	0.3	2.2
14	2.1	2.5	1.7	2.8	1.6	1.5	2.1	3.10	1.7	2.9	0.7	1.8	2.2	3.8	3.9	3.9	1.9	0.6	0.2
15	0.3	2.2	2.6	3.9	2.9	0.7	1.8	2.2	3.8	3.9	0.9	0.6	1.10	2.3	3.5	3.6	3.6	3.7	0.5
16	0.6	0.2	2.3	2.7	0.9	0.9	0.6	1.10	2.3	3.5	3.6	0.8	0.5	3.4	2.4	3.1	3.2	3.2	3.3
17	3.7	0.5	0.1	3.5	0.10	0.8	0.8	0.5	3.4	2.4	3.1	3.2	1.2	1.1	0.4	1.4	2.5	2.8	2.8
18	3.2	3.3	3.4	0.0	3.1	1.3	1.2	1.2	1.1	0.4	1.4	2.5	2.8	1.6	1.5	0.3	3.10	2.6	2.9
19	1.3	1.2	2.10	0.4	2.1	2.5	1.7	1.6	1.6	1.5	0.3	3.10	2.6	2.9	1.9	1.8	0.2	3.8	2.7
20	2.5	1.7	1.6	0.7	0.3	2.2	2.6	3.9	1.9	1.9	1.8	0.2	3.8	2.7	0.9	3.7	1.10	0.1	3.5
21	2.2	2.6	3.9	1.9	0.6	0.2	2.3	2.7	3.6	3.7	3.7	1.10	0.1	3.5	0.10	0.8	3.3	3.4	0.0
22	0.2	2.3	2.7	3.6	3.7	0.5	0.1	2.4	0.10	3.2	3.3	3.3	3.4	0.0	3.1	1.3	1.2	2.10	0.4
23	0.5	0.1	2.4	0.10	3.2	3.3	1.1	0.0	1.4	1.6	2.8	2.10	2.10	0.4	2.1	2.5	1.7	1.6	0.7
24	3.3	1.1	0.0	1.4	1.3	2.8	2.10	1.5	2.1	3.10	1.7	2.9	0.7	0.7	0.3	2.2	2.6	3.9	1.9
25	1.6	2.10	1.5	2.1	3.10	1.7	2.9	0.7	1.8	2.2	3.8	3.9	0.9	0.6	0.6	0.2	2.3	2.7	3.6
26	2.9	1.9	0.7	1.8	2.2	0.9	3.9	0.9	0.6	1.10	2.3	3.5	3.6	0.8	0.5	0.5	0.1	2.4	0.10
27	2.7	0.9	3.7	0.6	1.10	2.3	0.8	3.6	0.8	0.5	3.4	2.4	3.1	3.2	1.2	1.1	1.1	0.0	1.4
28	3.5	0.10	0.8	3.3	0.5	3.4	2.4	1.2	3.2	1.2	1.1	0.4	1.4	2.5	2.8	1.6	1.5	1.5	2.1
29	0.0	3.1	1.3	1.2	2.10	1.1	0.4	1.4	1.6	2.8	1.6	1.5	0.3	3.10	2.6	2.9	1.9	1.8	1.8
30	0.4	2.1	2.5	1.7	1.6	0.7	1.5	0.3	3.10	1.9	2.9	1.9	1.8	0.2	3.8	2.7	0.9	3.7	1.10
31	0.7	0.3	2.2	2.6	3.9	1.9	0.6	1.8	0.2	3.8	3.7	0.9	3.7	1.10	0.1	3.5	0.10	0.8	3.3
32	1.9	0.6	0.2	2.3	2.7	3.6	3.7	0.5	1.10	0.1	3.5	0.10	0.8	3.3	3.4	0.0	3.1	1.3	1.2
33	3.6	3.7	0.5	0.1	2.4	0.10	3.2	3.3	1.1	3.4	0.0	3.1	1.3	1.2	2.10	0.4	2.1	2.5	1.7
34	0.10	3.2	3.3	1.1	0.0	1.4	1.3	2.8	2.10	1.5	0.4	2.1	2.5	1.7	1.6	0.7	0.3	2.2	2.6
35	1.4	1.3	2.8	2.10	1.5	2.1	3.10	1.7	2.9	0.7	1.8	0.3	2.2	2.6	3.9	1.9	0.6	0.2	2.3
36	2.1	3.10	1.7	2.9	0.7	1.8	2.2	3.8	3.9	0.9	0.6	1.10	0.2	2.3	2.7	3.6	3.7	0.5	0.1

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